1. (30 pts) Parts (a) and (b) are not related.

   (a) Find the most general form of $u(x)$ such that $u''(x) = 3x^{-6} + 2x^{4/3} - \cos x + 8$. Fully simplify all fractions in your result.

   Solution:

$$
\begin{aligned}
u''(x) &= 3x^{-6} + 2x^{4/3} - \cos x + 8 \\
u'(x) &= 3 \cdot \frac{x^{-5}}{-5} + 2 \cdot \frac{x^{7/3}}{7/3} - \sin x + 8x + C_1 \\
&= -\frac{3}{5}\,x^{-5} + \frac{6}{7}\,x^{7/3} - \sin x + 8x + C_1 \\
u(x) &= -\frac{3}{5} \cdot \frac{x^{-4}}{-4} + \frac{6}{7} \cdot \frac{x^{10/3}}{10/3} + \cos x + 4x^2 + C_1 x + C_2 \\
&= \boxed{\frac{3}{20}\,x^{-4} + \frac{9}{35}\,x^{10/3} + \cos x + 4x^2 + C_1 x + C_2}
\end{aligned}
$$

(b) Consider a particle that is moving along a linear path with an acceleration of $a(t) = 6 - 12t$ m/s^2, where $t \geq 0$. Suppose the particle's initial velocity is 12 m/s and its inital position is 5 meters.

i. At what time will the particle be at rest? Include the correct unit of measurement.

ii. What is the particle's position when it is at rest? Include the correct unit of measurement.

Solution:

i. The particle's velocity, $v(t)$, is the antiderivative of $a(t)$ that satisfies $v(0) = 12$.

$$\begin{aligned} a(t) &= -12t + 6 \\ v(t) &= -6t^2 + 6t + C_1 \\ v(0) &= C_1 = 12 \\ v(t) &= -6t^2 + 6t + 12 \\ 0 &= -6t^2 + 6t + 12 = -6(t^2 - t - 2) = -6(t-2)(t+1) \\ t &= -1, 2 \end{aligned}$$

Since the given domain for t is $[0, \infty)$, the particle is only at rest at $\boxed{t = 2 \text{ seconds}}$

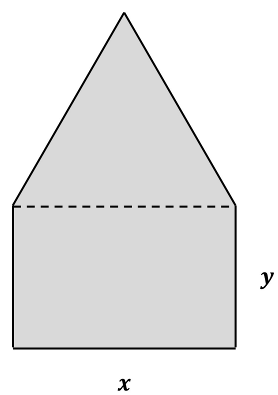
ii. The particle's position, $s(t)$, is the antiderivative of $v(t)$ that satisfies $s(0) = 5$.

$$\begin{aligned} v(t) &= -6t^2 + 6t + 12 \\ s(t) &= -2t^3 + 3t^2 + 12t + C_2 \\ s(0) &= C_2 = 5 \\ s(t) &= -2t^3 + 3t^2 + 12t + 5 \\ s(2) &= (-2)(2^3) + (3)(2^2) + (12)(2) + 5 = -16 + 12 + 24 + 5 = 25 \end{aligned}$$

Therefore, the particle's position when it is at rest is $\boxed{s = 25 \text{ meters}}$

2. (20 pts) The shaded region depicted below consists of a rectangle and an equilateral triangle positioned adjacent to each other, as drawn. The variables x and y represent the dimensions of the rectangle. If the perimeter of the entire shaded region (which consists of the five solid lines and does not include the dashed line) is 10 inches, determine the value of x (including the correct unit of measurement) that maximizes the area of the shaded region. Use the Second Derivative Test to confirm that your result is a local maximum value of the area function.

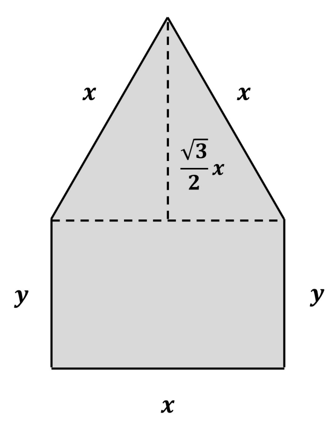
Note that the height of an equilateral triangle with a base length of x is $\dfrac{\sqrt{3}}{2}\,x$.



Solution:

Since the dashed line in the diagram referred to in the problem statement is the side of the rectangle that is opposite the side of length x, the length of the dashed line is also x. Since the dashed line is also one of the sides of the equilateral triangle, the other two sides of the triangle are also of length x.

The following diagram incorporates the preceding information as well as the information that is provided in the note below the problem statement.



The following is the constraint equation, which follows from the given statement that the perimeter of the shaded area equals 10 inches:

$$3x + 2y = 10$$

The objective function is the expression for the area of the shaded region because that is the quantity that is to be maximized. The area of the shaded region, denoted by A, equals the area of the rectangle plus the area of the triangle.

The area of the rectangle is length times width, which equals xy. The area of the triangle is one half base times height, where the base of the triangle is x and the height of the triangle is $\sqrt{3}x/2$. Therefore, the area of the shaded region is:

$$\begin{aligned} A &= xy + \frac{1}{2} \cdot x \cdot \frac{\sqrt{3}x}{2} \\ &= xy + \frac{\sqrt{3}}{4}\, x^2 \end{aligned}$$

The constraint equation implies that $y = \frac{1}{2}\,(10-3x)$, which can be substituted into the expression for A to produce the following objective function:

$$\begin{aligned} A(x) &= x\left(\frac{1}{2}\,(10-3x)\right) + \frac{\sqrt{3}}{4}\, x^2 \\ &= 5x - \frac{3}{2}\, x^2 + \frac{\sqrt{3}}{4}\, x^2 \end{aligned}$$

To find the critical numbers of $A(x)$, set $A'(x) = 0$:

$$\begin{aligned} A'(x) &= 5 - 3x + \frac{\sqrt{3}}{2}\, x \\ &= 5 - \frac{1}{2}\,(6-\sqrt{3})x = 0 \end{aligned}$$

Therefore, $\boxed{x = \dfrac{10}{6-\sqrt{3}} \text{ inches}}$

Note that $\boxed{A''(x) = -\frac{1}{2}\,(6-\sqrt{3}) < 0}$ for all values of x. Therefore, the Second Derivative Test confirms that there is a local maximum value of $A(x)$ at $x = \dfrac{10}{6-\sqrt{3}}$.

3. (20 pts) Suppose Newton's Method is used to estimate the value of the root of $y = r(x) = \dfrac{2-x}{x^2}$.

(a) Determine the value of x_1 produced by Newton's Method for an initial value of $x_0 = 1$.

(b) Would Newton's Method converge to a solution if the initial value was $x_0 = 4$? Explain why or why not.

(c) Write the general expression for Newton's Method for the specified function $r(x)$. Your answer should be an expression for x_{n+1} in terms of x_n.

Solution:

(a) When using Newton's Method, we have $x_1 = x_0 - \dfrac{r(x_0)}{r'(x_0)}$.

$$r'(x) = \frac{x^2(-1) - (2-x)(2x)}{x^4} = \frac{-x^2 - 4x + 2x^2}{x^4} = \frac{x^2 - 4x}{x^4} = \frac{x-4}{x^3}$$

$$r(1) = \frac{2-1}{1^2} = 1$$

$$r'(1) = \frac{1-4}{1^3} = -3$$

Therefore, $x_1 = 1 - \dfrac{1}{-3} = \boxed{x_1 = \dfrac{4}{3}}$

(b) Newton's Method would $\boxed{\text{not converge}}$ for $x_0 = 4$ because $r'(4) = \dfrac{4-4}{4^3} = 0$, which leads to division by zero when computing x_1.

(c) The general expression for Newton's Method is $x_{n+1} = x_n - \dfrac{r(x_n)}{r'(x_n)}$.

Therefore, Newton's Method can be expressed as follows for $r(x) = \dfrac{2-x}{x^2}$:

$$x_{n+1} = x_n - r(x_n) \cdot \frac{1}{r'(x_n)} = x_n - \frac{2-x_n}{x_n^2} \cdot \frac{x_n^3}{x_n - 4}$$

$$\boxed{x_{n+1} = x_n - \frac{x_n(2-x_n)}{x_n - 4}}$$

4. (30 pts) Parts (a) and (b) are not related.

   (a) Let $f(x) = x^{2/3}(1-x)$.

      i. Identify all critical numbers of $f(x)$.

      ii. For which values of x is $f(x)$ increasing and for which values of x is $f(x)$ decreasing? Express your answers using interval notation.

      iii. Identify the x-coordinate of each local maximum and minimum of $f(x)$, if any. Use the First Derivative Test to classify each one.

Solution:

i.

$$\begin{aligned} f(x) &= x^{2/3} - x^{5/3} \\ f'(x) &= \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} \\ &= \frac{1}{3}x^{-1/3}(2-5x) \\ &= \frac{1}{3\sqrt[3]{x}}(2-5x) \end{aligned}$$

$f' = 0$ at $x = 2/5$

f' does not exist at $x = 0$, and $x = 0$ is in the domain of f

Therefore, f has two critical numbers: $\boxed{0, 2/5}$

ii. The following table indicates that $f' < 0$ on $(-\infty, 0) \cup (2/5, \infty)$ and $f' > 0$ on $(0, 2/5)$.

	$x < 0$	$0 < x < \frac{2}{5}$	$x > \frac{2}{5}$
$2-5x$	+	+	−
$\sqrt[3]{x}$	−	+	+
$f'(x) = \dfrac{2-5x}{3\sqrt[3]{x}}$	−	+	−

Dividing points on the x-axis: $x = 0$, $x = \frac{2}{5}$

Therefore, $\boxed{f \text{ is decreasing on } (-\infty, 0) \cup (2/5, \infty)}$ and $\boxed{f \text{ is increasing on } (0, 2/5)}$

iii. f is continuous at $x = 0$ and f transitions from decreasing to increasing at that location. Therefore, $\boxed{f \text{ has a local minimum at } x = 0}$

f is continuous at $x = 2/5$ and f transitions from increasing to decreasing at that location. Therefore, $\boxed{f \text{ has a local maximum at } x = 2/5}$

(b) Let $g(x) = \frac{x^2}{4} - \cos x$.

i. For which values of x on the interval $(0, 2\pi)$ is $g(x)$ concave up and for which values of x on that same interval is $g(x)$ concave down? Express your answers using interval notation.

ii. Identify the x-coordinate of each inflection point of $g(x)$ on the interval $(0, 2\pi)$, if any. Justify your answer.

Solution:

i.

$$g'(x) = \frac{x}{2} + \sin x$$

$$g''(x) = \frac{1}{2} + \cos x$$

Next, identify values of x on $(0, 2\pi)$ such that $g'' = 0$.

$$\frac{1}{2} + \cos x = 0$$

$$\cos x = -\frac{1}{2}$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Since $g''(x)$ is continuous on the interval $(0, 2\pi)$ and there are exactly two values of x on that interval such that $g''(x) = 0$, then the sign of $g''(x)$ does not change anywhere on each of the following subintervals: $(0, 2\pi/3)$, $(2\pi/3, 4\pi/3)$, and $(4\pi/3, 2\pi)$. Therefore, determination of the sign of $g''(x)$ at any x value on each subinterval serves to determine the sign of $g''(x)$ everyone on that subinterval.

$$(0, 2\pi/3): \quad g''(\pi/2) = 1/2 + \cos(\pi/2) = 1/2 > 0$$

$$(2\pi/3, 4\pi/3): \quad g''(\pi) = 1/2 + \cos(\pi) = 1/2 - 1 < 0$$

$$(4\pi/3, 2\pi): \quad g''(3\pi/2) = 1/2 + \cos(3\pi/2) = 1/2 > 0$$

Therefore, $\boxed{g \text{ is concave up on } (0, 2\pi/3) \cup (4\pi/3, 2\pi)}$ and $\boxed{g \text{ is concave down on } (2\pi/3, 4\pi/3)}$

ii. Note that g is continuous on $(0, 2\pi)$, and in particular at $x = 2\pi/3$ and $x = 4\pi/3$.

Since g changes concavity at $x = 2\pi/3$ and at $x = 4\pi/3$, $g(x)$ has inflection points at

$\boxed{x = 2\pi/3 \text{ and at } x = 4\pi/3}$