Write your name below. This exam is worth 150 points. You must show all your work to receive credit on each problem and must fully simplify your answers unless otherwise instructed. You are allowed to use one page of notes. You are not allowed to use a calculator or any computational software.

Name:

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1. (15 points) An out of control rocket is traveling along the space curve given by $\mathbf{r}(t) = (5t + t^2)\mathbf{i} + 2\cos\left(\frac{\pi t}{3}\right)\mathbf{j} + \left(8 + 2\sin\left(\frac{\pi t}{3}\right)\right)\mathbf{k}$.

   (a) (6 points) Find the velocity of the rocket when $t = 5$.

   Solution:

$$\mathbf{r}'(t) = (5 + 2t)\mathbf{i} - \frac{2\pi}{3}\sin\left(\frac{\pi t}{3}\right)\mathbf{j} + \frac{2\pi}{3}\cos\left(\frac{\pi t}{3}\right)\mathbf{k}$$
$$\mathbf{r}'(5) = 15\mathbf{i} - \frac{2\pi}{3}\sin\left(\frac{5\pi}{3}\right)\mathbf{j} + \frac{2\pi}{3}\cos\left(\frac{5\pi}{3}\right)\mathbf{k}$$
$$\mathbf{r}'(5) = 15\mathbf{i} - \frac{2\pi}{3}\left(-\frac{\sqrt{3}}{2}\right)\mathbf{j} + \frac{2\pi}{3}\left(\frac{1}{2}\right)\mathbf{k}$$
$$\mathbf{r}'(5) = 15\mathbf{i} + \frac{\pi}{\sqrt{3}}\mathbf{j} + \frac{\pi}{3}\mathbf{k}$$

   (b) (9 points) Assuming that the rocket exhaust is ejected in the opposite direction of the rocket's velocity, where does the exhaust ejected at $t = 5$ intersect the xy-plane?

   Solution: The exhaust travels along the line $\mathbf{r}(5) - \mathbf{r}'(5)s$, with parameter s.

$$\mathbf{r}(5) = (5 \cdot 5 + 5^2)\mathbf{i} + 2\cos\left(\frac{5\pi}{3}\right)\mathbf{j} + \left(8 + 2\sin\left(\frac{5\pi}{3}\right)\right)\mathbf{k}$$
$$\mathbf{r}(5) = 50\mathbf{i} + 2\left(\frac{1}{2}\right)\mathbf{j} + \left(8 + 2\left(-\frac{\sqrt{3}}{2}\right)\right)\mathbf{k}$$
$$\mathbf{r}(5) = 50\mathbf{i} + \mathbf{j} + \left(8 - \sqrt{3}\right)\mathbf{k}$$

   so our new line is

$$\begin{aligned}\mathbf{r}(5) - \mathbf{r}'(5)s &= 50\mathbf{i} + \mathbf{j} + \left(8 - \sqrt{3}\right)\mathbf{k} - \left(15\mathbf{i} + \frac{\pi}{\sqrt{3}}\mathbf{j} + \frac{\pi}{3}\mathbf{k}\right)s \\ &= (50 - 15s)\,\mathbf{i} + \left(1 - \frac{\pi}{\sqrt{3}}s\right)\mathbf{j} + \left(8 - \sqrt{3} - \frac{\pi}{3}s\right)\mathbf{k}\end{aligned}$$

   The intersection with the xy-plane happens when the z component equals 0, which allows us to calculate s:

$$\begin{aligned}8 - \sqrt{3} - \frac{\pi}{3}s &= 0 \\ \frac{\pi}{3}s &= 8 - \sqrt{3} \\ s &= 3\frac{8 - \sqrt{3}}{\pi}\end{aligned}$$

Substituting s into our line gives

$$\begin{aligned}
\mathbf{r}(5) - \mathbf{r}'(5) \cdot 3\frac{8-\sqrt{3}}{\pi} &= \left(50 - 15 \cdot 3\frac{8-\sqrt{3}}{\pi}\right)\mathbf{i} + \left(1 - \frac{\pi}{\sqrt{3}} 3\frac{8-\sqrt{3}}{\pi}\right)\mathbf{j} \\
&= \left(50 - 15 \cdot 3\frac{8-\sqrt{3}}{\pi}\right)\mathbf{i} + \left(1 - \sqrt{3}(8-\sqrt{3})\right)\mathbf{j} \\
&= \left(50 - 15 \cdot 3\frac{8-\sqrt{3}}{\pi}\right)\mathbf{i} + \left(4 - 8\sqrt{3})\right)\mathbf{j} \\
&= \left(\frac{50\pi - 340 + 45\sqrt{3}}{\pi}\right)\mathbf{i} + \left(4 - 8\sqrt{3})\right)\mathbf{j}
\end{aligned}$$

2. (20 points) The following two problems are unrelated.

   (a) (6 points) Find the following limit or prove that it does not exist:

   $$\lim_{(x,y)\to(0,1)} \frac{x - xy}{x^2 - (y-1)^2}$$

   Solution: This limit does not exist. If we approach the point $(0, 1)$ along a line with slope m, $y = mx + 1$, we find that

   $$\begin{aligned} \frac{x - xy}{x^2 - (y-1)^2} &\to \frac{x - x(mx+1)}{x^2 - (mx+1-1)^2} \\ &= \frac{x - mx^2 - x}{x^2 - (mx)^2} \\ &= \frac{-mx^2}{x^2(1-m^2)} \\ &= -\frac{m}{1-m^2} \end{aligned}$$

   As this depends on the slope of the line, the limit does not exist.

   (b) (14 points) Find the degree two Taylor approximation of $f(x, t) = \sin(3x - 4t)$ near the point $(0, 0)$.

   Solution: We must calculate all the partial derivatives to second degree at $(0, 0)$ then use the Taylor approximation formula:

   $$\begin{aligned} f(x,t) &= \sin(3x - 4t) & f(0,0) &= 0 \\ f_x(x,t) &= 3\cos(3x - 4t) & f_x(0,0) &= 3 \\ f_t(x,t) &= -4\cos(3x - 4t) & f_x(0,0) &= -4 \\ f_{xx}(x,t) &= -9\sin(3x - 4t) & f_x x(0,0) &= 0 \\ f_{xt}(x,t) &= 12\sin(3x - 4t) & f_x t(0,0) &= 0 \\ f_{tt}(x,t) &= 16\sin(3x - 4t) & f_t t(0,0) &= 0 \end{aligned}$$

   Bringing this together gives us our degree two approximation:

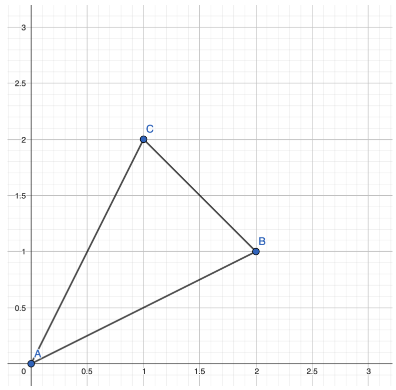
   $$Q(x, t) = 3x - 4t$$

3. (20 points) Let D be the triangular region in the xy-plane with vertices $(0,0)$, $(2,1)$, $(1,2)$ and f be the function $f(x,y) = x^2 + 2xy + y^2$ and T be the coordinate transformation given by

$$u(x,y) = x + y$$
$$v(x,y) = 2x - y$$

(a) (6 points) Set up but do not evaluate the integral of f over D in the xy-plane.

Solution: Our original region D:



The lines which make the sides of the triangle are

$$y = 2x$$
$$y = \frac{x}{2}$$
$$y = 3 - x$$

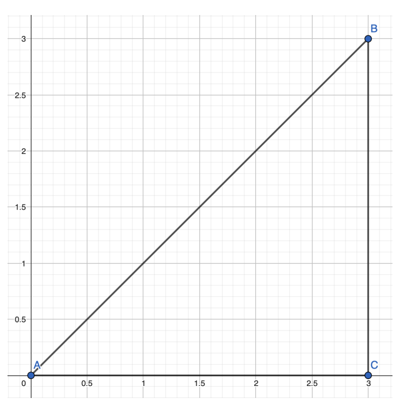
so our integral is

$$\int_0^1 \int_{x/2}^{2x} (x^2 + 2xy + y^2)\,dydx + \int_1^2 \int_{x/2}^{3-x} (x^2 + 2xy + y^2)\,dydx$$

(b) (6 points) Sketch the image of D in the uv-plane, labeling all relevant points.

Solution: Our image is also a triangle. The vertices change to

$$(0,0) \to (0,0)$$
$$(2,1) \to (3,3)$$
$$(1,2) \to (3,0)$$



(c) (8 points) Set up but do not evaluate the integral of f over the image of D in the uv-plane.

Solution: We have to convert f and find the Jacobian:

$$\begin{aligned} f(x,y) &= x^2 + 2xy + y^2 \\ &= (x+y)^2 \\ &= u^2 \end{aligned}$$

For the Jacobian we must find the inverse transformation first:

$$\begin{aligned} u + v = x + y + 2x - y &= 3x \\ x &= \frac{u+v}{3} \\ v &= 2x - y \\ y = 2x - v &= 2\frac{u+v}{3} - v \\ y &= \frac{2u-v}{3} \end{aligned}$$

Then find the determinate:

$$\begin{aligned} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} &= \begin{vmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{vmatrix} \\ &= \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \\ &= -\frac{1}{3} \end{aligned}$$

So the Jacobian is

$$J = \left|-\frac{1}{3}\right| = \frac{1}{3}$$

Using the image of D as the integration region gives us the integral:

$$\int_0^3 \int_0^u \frac{u^2}{3} dvdu$$

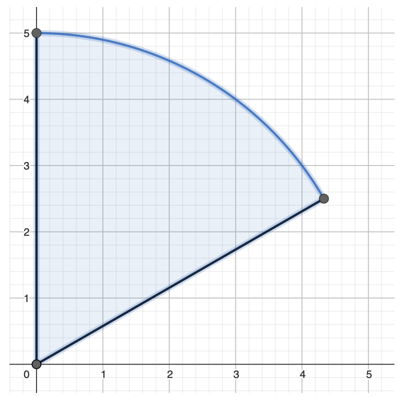
or

$$\int_0^3 \int_v^3 \frac{u^2}{3} dudv$$

4. (20 points) The Pteranodon enclosure for the local Cretaceous Garden theme park consists of the volume bounded on top by a sphere centered on the origin and below by the upper half of a cone, again centered at the origin. The intersection of these surfaces occurs at $r = \dfrac{5\sqrt{3}}{2}$ and $z = \dfrac{5}{2}$ in cylindrical coordinates. As Pteranodons prefer to fly high, the density of Pteranodons is given by $f(x, y, z) = 10z - x^2 - y^2$ in rectangular coordinates.

   (a) (8 points) Set up but do not evaluate the integral of f over the enclosure in cylindrical coordinates.

   Solution: A constant θ trace in the rz-plane:



   The equation for the line from the origin to the intersection is $z = \frac{1}{\sqrt{3}}r$, while the sphere has a radius of 5. Our integral is

$$\int_0^{2\pi} \int_0^{\frac{5\sqrt{3}}{2}} \int_{\frac{1}{\sqrt{3}}r}^{\sqrt{25-r^2}} (10z - r^2) r\,dz\,dr\,d\theta$$

   (b) (12 points) Set up but do not evaluate the integral of f over the enclosure in rectangular coordinates.

   Solution: The region projected into the xy-plane is a circle with radius $\frac{5\sqrt{3}}{2}$, so our integral is

$$\int_{-\frac{5\sqrt{3}}{2}}^{\frac{5\sqrt{3}}{2}} \int_{-\sqrt{\frac{75}{4}-x^2}}^{\sqrt{\frac{75}{4}-x^2}} \int_{\sqrt{\frac{x^2+y^2}{3}}}^{\sqrt{25-x^2-y^2}} (10z - x^2 - y^2)\,dz\,dy\,dx$$

5. (22 points) Given the following system of linear equations

$$\begin{aligned} 2x_1 - 4x_2 + 6x_4 &= 2 \\ -x_1 + 2x_2 + x_3 - 2x_4 &= -2 \\ 2x_1 - 4x_2 - x_3 + 5x_4 &= 3 \end{aligned}$$

(a) (10 points) Find the RREF of the augmented matrix for the system.
Solution:

$$\left(\begin{array}{cccc|c} 2 & -4 & 0 & 6 & 2 \\ -1 & 2 & 1 & -2 & -2 \\ 2 & -4 & -1 & 5 & 3 \end{array}\right) \xrightarrow[R_1=\frac{1}{2}R_1]{} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ -1 & 2 & 1 & -2 & -2 \\ 2 & -4 & -1 & 5 & 3 \end{array}\right)$$

$$\xrightarrow[R_3=R_3-2R_1]{R_2=R_2+R_1} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[R_3=R_3+R_2]{} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(b) (4 points) Identify the independent and dependent variables of the system.

Solution: The independent (or free) variables are x_2 and x_4. The dependent variables are x_1 and x_3.

(c) (8 points) Find the general solution for the system.

Solution: Let $x_2 = s$ and $x_4 = t$, then the second row tells us that

$$\begin{aligned} x_3 + t &= -1 \\ x_3 &= -1 - t \end{aligned}$$

and the first row tells us that

$$\begin{aligned} x_1 - 2s + 3t &= 1 \\ x_1 &= 1 + 2s - 3t \end{aligned}$$

so the general solutions is

$$\mathbf{x} = \begin{pmatrix} 1 + 2s - 3t \\ s \\ -1 - t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix} t$$

6. (15 points) Let $A = \begin{pmatrix} 1 & a \\ a & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. If possible, find values of a where the equation $A\mathbf{x} = \mathbf{b}$ has

   (a) No solutions.

   (b) One unique solution.

   (c) Infinite solutions.

   If no such value of a exists, write 'Not possible.'

   Solution: We first need to row reduce the augmented matrix into REF form:

$$\left(\begin{array}{cc|c} 1 & a & 3 \\ a & 2 & 4 \end{array}\right) \xrightarrow[R_2 = R_2 - a \cdot R_1]{} \left(\begin{array}{cc|c} 1 & a & 3 \\ 0 & 2 - a^2 & 4 - 3a \end{array}\right)$$

   (a) To check for no solutions, we set the bottom right entry equal to 0 and solve for a:

$$\begin{aligned} 2 - a^2 &= 0 \\ a &= \pm\sqrt{2} \end{aligned}$$

   Substituting these values of a into the augmented matrix gives two systems:

$$\left(\begin{array}{cc|c} 1 & \pm\sqrt{2} & 3 \\ 0 & 0 & 4 \mp 3\sqrt{2} \end{array}\right)$$

   neither of which has solutions since $4 \mp 3\sqrt{2} \neq 0$.

   (b) For a system with a unique solution, we choose any value of $a \neq \pm\sqrt{2}$, for example when $a = 1$ we get

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right)$$

   which has the unique solution $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

   (c) It is impossible to have a row completely of 0s, since If $4 - 3a = 0$ then $2 - a^2 \neq 0$ and vice versa. We conclude that it is not possible to have infinite solutions to these systems.

7. (15 points) Let A be the matrix given by the product:

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}$$

Use the properties of matrix inverses to find A^{-1} without calculating A.

Solution: The inverse of a product is the product of the inverses in the reversed order:

$$A^{-1} = \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}^{-1}$$

$$= \frac{1}{3-4} \begin{pmatrix} 3 & -4 \\ -1 & 1 \end{pmatrix} \frac{1}{3-2} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -11 & 15 \\ 3 & -4 \end{pmatrix}$$

8. (23 points) Given the points $(-2,-2)$, $(-1,1)$, $(0,2)$, and $(0,3)$

   (a) (16 points) Find the least squares best fit line through the points.

   Solution: Our model is $y = c_0 + c_1 x$. We create the coefficient matrix and vector from the datapoints, getting:

$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

   We find the solution to $A^T A\mathbf{v} = A^T\mathbf{b}$:

$$A^T A = \begin{pmatrix} 4 & -3 \\ -3 & 5 \end{pmatrix}$$
$$A^T\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

   So we solve the augmented matrix

$$\begin{aligned} &\left(\begin{array}{cc|c} 4 & -3 & 4 \\ -3 & 5 & 3 \end{array}\right) \\ \xrightarrow[R_2 = R_2 + \frac{3}{4}R_1]{} &\left(\begin{array}{cc|c} 4 & -3 & 4 \\ 0 & 11/4 & 6 \end{array}\right) \\ \xrightarrow[R_2 = \frac{4}{11}R_2]{} &\left(\begin{array}{cc|c} 4 & -3 & 4 \\ 0 & 1 & 24/11 \end{array}\right) \\ \xrightarrow[R_1 = R_1 + 3R_2]{} &\left(\begin{array}{cc|c} 4 & 0 & 116/11 \\ 0 & 1 & 24/11 \end{array}\right) \\ \xrightarrow[R_1 = \frac{1}{4}R_1]{} &\left(\begin{array}{cc|c} 1 & 0 & 29/11 \\ 0 & 1 & 24/11 \end{array}\right) \end{aligned}$$

   Our best fit is therefore

$$y = \frac{29}{11} + \frac{24}{11}x$$

(b) (7 points) Calculate the error vector.

Solution:

$$\begin{aligned}
\mathbf{e} &= A\mathbf{v} - \mathbf{b} \\
&= \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 29/11 \\ 24/11 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 2 \\ 3 \end{pmatrix} \\
&= \frac{29}{11} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{24}{11} \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 2 \\ 3 \end{pmatrix} \\
&= \frac{1}{11} \begin{pmatrix} 29 \\ 29 \\ 29 \\ 29 \end{pmatrix} + \frac{1}{11} \begin{pmatrix} -48 \\ -24 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{11} \begin{pmatrix} -22 \\ 11 \\ 22 \\ 33 \end{pmatrix} \\
&= \frac{1}{11} \begin{pmatrix} 3 \\ -6 \\ 7 \\ -4 \end{pmatrix}
\end{aligned}$$