INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **show all work**. A correct answer with incorrect or no supporting work may receive no credit. Books, notes, electronic devices (such as calculator or other unauthorized electronic resources), help from another person, are not permitted during the exam. **Give all answers in exact form.**

1. The following are unrelated. (6 pts)

   (a) Perform the indicated operations: $e^0 + \frac{5}{12} - \frac{11}{18}$

   Solution:

$$\begin{aligned} e^0 + \frac{5}{12} - \frac{11}{18} &= 1 + \frac{5}{12} - \frac{11}{18} && (1) \\ &= \frac{36}{36} + \frac{15}{36} - \frac{22}{36} && (2) \\ &= \frac{36+15-22}{36} && (3) \\ &= \boxed{\frac{29}{36}} && (4) \end{aligned}$$

   (b) Evaluate the expression: $-2\sqrt{32} + \sqrt{50}$

   Solution:

$$\begin{aligned} -2\sqrt{32} + \sqrt{50} &= -2\sqrt{16 \cdot 2} + \sqrt{25 \cdot 2} && (5) \\ &= -2 \cdot 4\sqrt{2} + 5\sqrt{2} && (6) \\ &= (-8+5)\sqrt{2} && (7) \\ &= \boxed{-3\sqrt{2}} && (8) \end{aligned}$$

2. Rewrite each of the following without absolute value symbol (4 pts):

   (a) $|x-2|$ where $x > 2$

   Solution:

   Since $x > 2$ is true then we know that $x - 2 > 0$ as well, so the expression inside the absolute value is positive, and we can write:

$$|x-2| = \boxed{x-2} \qquad (9)$$

   (b) $|1-\sqrt{2}|$

   Solution:

   Since $\sqrt{2} > 1$, then we know $1 - \sqrt{2} < 0$, so the number inside the absolute value is negative, and we can multiply by -1 to make it positive:

$$\begin{aligned} |1-\sqrt{2}| &= -(1-\sqrt{2}) && (10) \\ &= \boxed{\sqrt{2}-1} && (11) \end{aligned}$$

3. Let a and b be real numbers such that $a \geq 0$ and $b < 0$. Determine whether the following expression is positive, negative, or the sign cannot be determined (2 pts):

$$2ab - 15b^2 3^b$$

Solution:

The first term is either negative or zero, depending on if a is zero or not:

$$2ab \leq 0 \tag{12}$$

The second term is always negative, since $b^2 > 0$ and $3^b > 0$:

$$-15b^2 3^b < 0 \tag{13}$$

The sum of two terms, one of which is always negative and the other which is negative or zero must be $\boxed{negative}$:

$$\boxed{2ab - 15b^2 3^b < 0} \tag{14}$$

4. The following are unrelated. (8 pts)

(a) Rewrite the expression with positive exponents and combine: $(a-1)^{-3}(a-1)^{-14}$

Solution:

$$\begin{aligned} (a-1)^{-3}(a-1)^{-14} &= (a-1)^{-17} &(15)\\ &= \boxed{\frac{1}{(a-1)^{17}}} &(16) \end{aligned}$$

(b) Perform the indicated operations: $x^4 x^{\frac{3}{2}} + \dfrac{3x^{-2}}{4\left(x^3\right)^{-\frac{5}{6}}}$

Solution:

$$\begin{aligned} x^4 x^{\frac{3}{2}} + \frac{3x^{-2}}{4\left(x^3\right)^{-\frac{5}{6}}} &= x^{\frac{11}{2}} + \frac{3\left(x^3\right)^{\frac{5}{6}}}{4x^2} &(17)\\ &= x^{\frac{11}{2}} + \frac{3x^{\frac{5}{2}}}{4x^2} &(18)\\ &= \boxed{x^{\frac{11}{2}} + \frac{3}{4}x^{\frac{1}{2}}} &(19) \end{aligned}$$

5. The following are unrelated. (8 pts)

(a) Find the sum, difference, product as indicated: $(2x-4)^2 - (x+2) + 2(1-3x)$

Solution:

$$\begin{aligned} (2x-4)^2 - (x+2) + 2(1-3x) &= (2x-4)(2x-4) - (x+2) + (2-6x) &(20)\\ &= (4x^2 - 8x - 8x + 16) - x - 2 + 2 - 6x &(21)\\ &= 4x^2 - 16x + 16 - 7x &(22)\\ &= \boxed{4x^2 - 23x + 16} &(23) \end{aligned}$$

(b) Multiply and simplify: $\left(5-\sqrt{1+x}\right)^2$

Solution:

$$\begin{aligned}
\left(5-\sqrt{1+x}\right)^2 &= \left(5-\sqrt{1+x}\right)\left(5-\sqrt{1+x}\right) && (24)\\
&= 25-5\sqrt{1+x}-5\sqrt{1+x}+\left(\sqrt{1+x}\right)^2 && (25)\\
&= 25-10\sqrt{1+x}+1+x && (26)\\
&= \boxed{26-10\sqrt{1+x}+x} && (27)
\end{aligned}$$

6. The following are unrelated. (9 pts)

(a) Perform the division and simplify: $\frac{\frac{3x^2-27}{x^2-6x+9}}{\frac{4x^2-16x}{2x-8}}$

Solution:

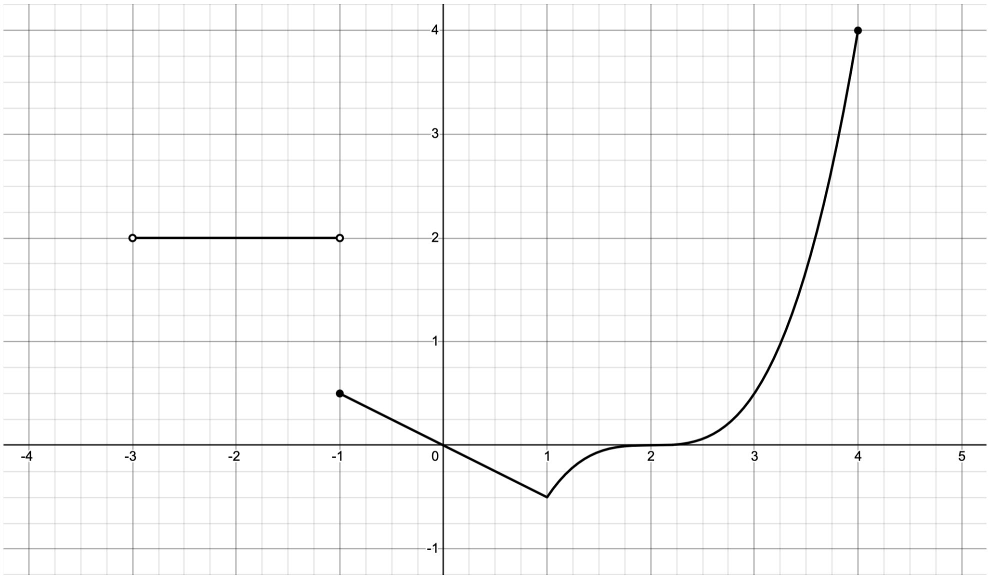
$$\begin{aligned}
\frac{\frac{3x^2-27}{x^2-6x+9}}{\frac{4x^2-16x}{2x-8}} &= \left(\frac{3x^2-27}{x^2-6x+9}\right)\left(\frac{2x-8}{4x^2-16x}\right) && (28)\\
&= \left(\frac{3(x^2-9)}{(x-3)(x-3)}\right)\left(\frac{2(x-4)}{4x(x-4)}\right) && (29)\\
&= \left(\frac{3(x-3)(x+3)}{(x-3)(x-3)}\right)\left(\frac{1}{2x}\right) && (30)\\
&= \left(\frac{3(x+3)}{(x-3)}\right)\left(\frac{1}{2x}\right) && (31)\\
&= \boxed{\frac{3(x+3)}{2x(x-3)}} && (32)
\end{aligned}$$

(b) Simplify the following: $\ln\left(e^3\right)+\log_3(3x)-\log_3(x)+\log_2(16)$ (Your answer should have no logarithms)

Solution:

$$\begin{aligned}
\ln\left(e^3\right)+\log_3(3x)-\log_3(x)+\log_2(16) &= 3+\log_3(3)+\log_3(x)-\log_3(x)+\log_2(2^4) && (33)\\
&= 3+1+4 && (34)\\
&= \boxed{8} && (35)
\end{aligned}$$

7. Use the graph of $y = f(x)$ below to answer the following (10 pts):



(a) Identify the domain of $f(x)$.

Solution: In interval notation: $\boxed{(-3, 4]}$

(b) Identify the range of $f(x)$.

Solution: In interval notation: $\boxed{\left[-\frac{1}{2}, 4\right]}$

(c) Solve the inequality $f(x) \geq \frac{1}{2}$. Give answer in interval notation.

Solution: $\boxed{(-3, -1] \cup [3, 4]}$

(d) If $h(x) = e^{2x}$ find $h(f(-2))$.

Solution: $f(-2) = 2$, so $g(f(-2)) = g(2) = e^{2 \cdot 2} = \boxed{e^4}$

(e) If $g(x) = 2^x - 1$ solve the equation $g(x) = f(x)$.

Solution: We can graph $g(x) = 2^x - 1$ on the graph above and see that the two functions only intersect at $(0, 0)$, so the solution is $\boxed{x = 0}$.

8. Use long division to find the **quotient** and **remainder** when $x^4 - 3x^3 - x^2$ is divided by $x^3 - 2x^2 + x$. (4 pts)

Solution:

$$\begin{array}{r|l} & \qquad\qquad\qquad x - 1 \\ x^3 - 2x^2 + x + 0 & x^4 - 3x^3 \quad - x^2 + 0x + 0 \\ & -x^4 + 2x^3 \quad - x^2 + 0x \\ \hline & \qquad -x^3 - 2x^2 + 0x + 0 \\ & \qquad\ \ x^3 - 2x^2 \quad + x + 0 \\ \hline & \qquad\qquad -4x^2 \quad + x + 0 \end{array}$$

The quotient is $\boxed{x - 1}$, and the remainder is $\boxed{-4x^2 + x}$.

9. Solve the following equations for the indicated variable. If there are no solutions, write **no solutions**. (8 pts)

(a) Solve for y: $\frac{3}{5}y - 1 = 2 + \frac{13}{10}y$

Solution:

$$\frac{3}{5}y - 1 = 2 + \frac{13}{10}y \tag{36}$$
$$\frac{3}{5}y - \frac{13}{10}y = 3 \tag{37}$$
$$-\frac{7}{10}y = 3 \tag{38}$$
$$y = \boxed{-\frac{30}{7}} \tag{39}$$

(b) Solve for x: $2x(x-6) = -32$

Solution:

$$2x(x-6) = -32 \tag{40}$$
$$2x^2 - 12x = -32 \tag{41}$$
$$2x^2 - 12x + 32 = 0 \tag{42}$$
$$2(x^2 - 6x + 16) = 0 \tag{43}$$
$$x^2 - 6x + 16 = 0 \tag{44}$$
$$\tag{45}$$

$x^2 - 6x + 16$ doesn't factor in the real numbers. Attempting the quadratic formula results in imaginary values so the answer is $\boxed{\textbf{no solutions}}$.

10. Solve for r when a is a constant and $a > 3$ (4 pts): $\log(3) = \log(2a) - \log(2+r)$

Solution:

$$\log(3) = \log(2a) - \log(2+r) \tag{46}$$
$$\log(3) = \log\left(\frac{2a}{2+r}\right) \tag{47}$$
$$3 = \frac{2a}{2+r} \tag{48}$$
$$3(2+r) = 2a \tag{49}$$
$$6 + 3r = 2a \tag{50}$$
$$3r = 2a - 6 \tag{51}$$
$$r = \boxed{\frac{2a-6}{3} = \frac{2}{3}a - 2} \tag{52}$$

11. If the recommended dosage of an adult drug is D (in mg), then to determine the appropriate dosage c for a child of age a (in years), pharmacists use the linear function $c = 0.12D(a+1)$. Suppose the dosage for an adult is 100 mg. (6 pts)

(a) What is the slope of the line (your answer should not include any variables)?

Solution: The slope is $0.12D = 0.12(100) = \boxed{12}$.

(b) What are the units of the slope?

Solution: $\boxed{\text{mg per year}}$; that is, the number of milligrams of dosage required per year of age.

(c) Find the average rate of change of the child dosage c from age 3 to age 9.

Solution:

$D = 100$, so the dosage c as a function of age a is $c(a) = 12(a+1)$. We wish to find the average value of $c(a)$ over the interval $[3, 9]$.

$$\begin{aligned} \text{average rate of change} &= \frac{c(9) - c(3)}{9 - 3} && (53) \\ &= \frac{12(9+1) - 12(3+1)}{6} && (54) \\ &= \frac{120 - 48}{6} && (55) \\ &= \frac{72}{6} && (56) \\ &= \boxed{12 \text{ mg/year}} && (57) \end{aligned}$$

12. Consider the functions: $h(x) = \sin(x)$ and $k(x) = \dfrac{3}{x}$. (8 pts)

(a) Fill in the blanks for the function $k(x)$: $k(x) \to$ ____ as $x \to -\infty$ and $k(x) \to$ ____ as $x \to \infty$.

Solution: $k(x) \to \boxed{0}$ as $x \to -\infty$ and $k(x) \to \boxed{0}$ as $x \to \infty$.

(b) Find $(h \circ k)(x)$.

Solution: $(h \circ k)(x) = h(k(x)) = \boxed{\sin\left(\frac{3}{x}\right)}$

(c) Find the domain of $(h \circ k)(x)$. Give your answer in interval notation.

Solution: $\boxed{(-\infty, 0) \cup (0, \infty)}$

13. For the rational function $r(x) = \dfrac{x-3}{3x^2 - 6x - 9}$ answer the following (11 pts):

(a) Find the domain of $r(x)$.

Solution:

We must find and exclude all values of x that make the denominator equal to zero.

$$\begin{aligned} \frac{x-3}{3x^2 - 6x - 9} &= \frac{x-3}{3(x^2 - 2x - 3)} \\ &= \frac{x-3}{3(x-3)(x+1)} \end{aligned}$$

The denominator is zero when $3(x-3)(x+1) = 0$, which has solutions $x = 3, -1$. Therefore the domain is $\boxed{(-\infty, -1) \cup (-1, 3) \cup (3, \infty)}$.

(b) Find the x-coordinate of any hole(s). If there are no hole(s) write NONE.

Solution:

From above,

$$\begin{aligned} r(x) &= \frac{x-3}{3(x-3)(x+1)} && (58) \\ &= \frac{1}{3(x+1)}, \ x \neq 3 && (59) \end{aligned}$$

Notice how the $x-3$ terms cancel out. This means that there is a hole where $x-3=0$, or $\boxed{x=3}$.

(c) Find the y-coordinate of any hole(s). If there are no hole(s) write NONE.

Solution:

We find the y-coordinate by plugging in $x=3$ to the reduced version of the function:

$$y_{\text{hole}} = \frac{1}{3(3+1)} = \boxed{\frac{1}{12}} \qquad (60)$$

Therefore the function has a hole at $\boxed{\left(3, \frac{1}{12}\right)}$.

(d) Identify the horizontal or slant asymptote of $r(x)$. If there is no horizontal or slant asymptote write NONE.

Solution:

Taking the ratio of the leading terms we get: $\frac{x}{3x^2} = \frac{1}{3x}$ which goes to zero as $x \to \pm\infty$ so there is a horizontal asymptote of $\boxed{y=0}$.

(e) Find all vertical asymptote(s). If there are none write NONE.

Solution:

The denominator is equal to zero when $x=3$ and $x=-1$. From above, there is a hole at $x=3$. However, there is no hole at $x=-1$: here, the function is undefined (the numerator is nonzero and the denominator is zero). Therefore there is a vertical asymptote at $\boxed{x=-1}$.

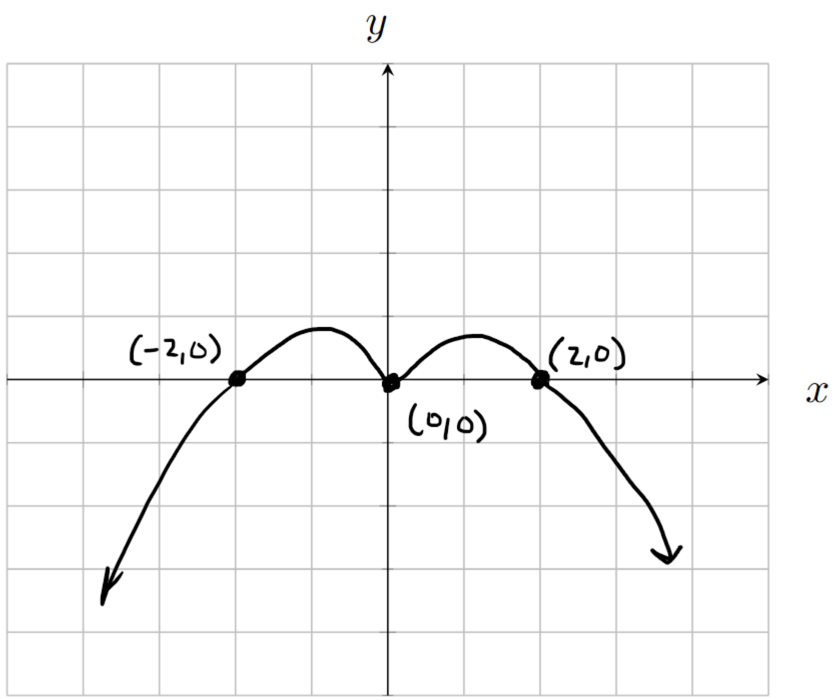
14. Answer the following for the polynomial function $y = P(x)$ with the following properties. (8 pts)

i. The graph of $y = P(x)$ touches (does not cross) the x-intercept $(0, 0)$.
ii. The graph of $y = P(x)$ crosses the x-intercept $(-2, 0)$.
iii. The graph of $y = P(x)$ satisfies: $P(x) \to -\infty$ as $x \to -\infty$.
iv. $y = P(x)$ is an even function.
v. $y = P(x)$ has one, and only one, additional x-intercept to those listed above.

(a) Sketch the graph of $y = P(x)$ that satisfies all the given properties. **Label all intercepts.**

Solution:

First label the intercepts at $(0, 0)$ and $(-2, 0)$. Since P is even (iv), it also cross at the x-intercept $(2, 0)$. By (v), there are no more intercepts.
We can also use the fact that P is even to conclude that $P(x) \to -\infty$ as $x \to \infty$, and that the graph should be symmetric about the y-axis. Using the end behavior and symmetry, we can fill in the shape of the graph.



(b) Find the equation of a degree 4 polynomial function for $P(x)$ that satisfies all of the given properties.

Solution:

The roots at $(2, 0)$ and $(-2, 0)$ must have multiplicity one, while the root at $(0, 0)$ must have multiplicity two. Because of the end behavior, we know that the leading coefficient of the polynomial must be negative. No other information is given. Therefore, any poynomial of the form

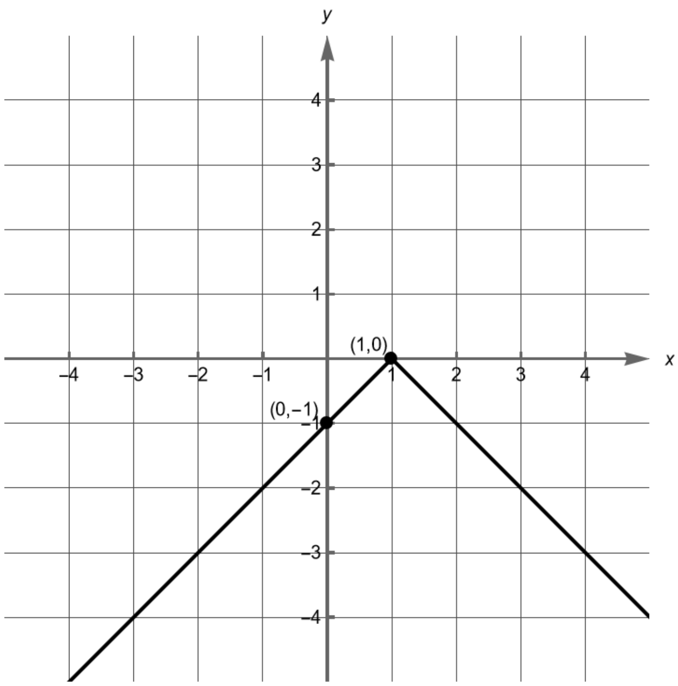
$$P(x) = -ax^2(x - 2)(x + 2)$$

is a valid solution, where a can be any positive real number. Specifically, $\boxed{P(x) = -x^2(x - 2)(x + 2)}$ is a reasonable answer.

15. Sketch the graph of the following functions. **Label all intercepts and asymptotes as appropriate.** (15 pts)

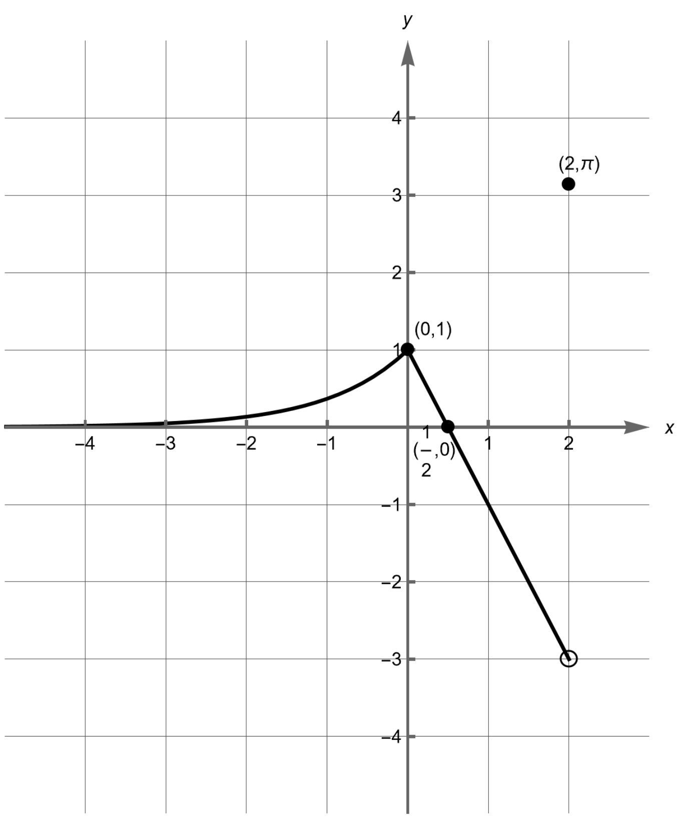
(a) $g(x) = -|x-1|$

Solution:



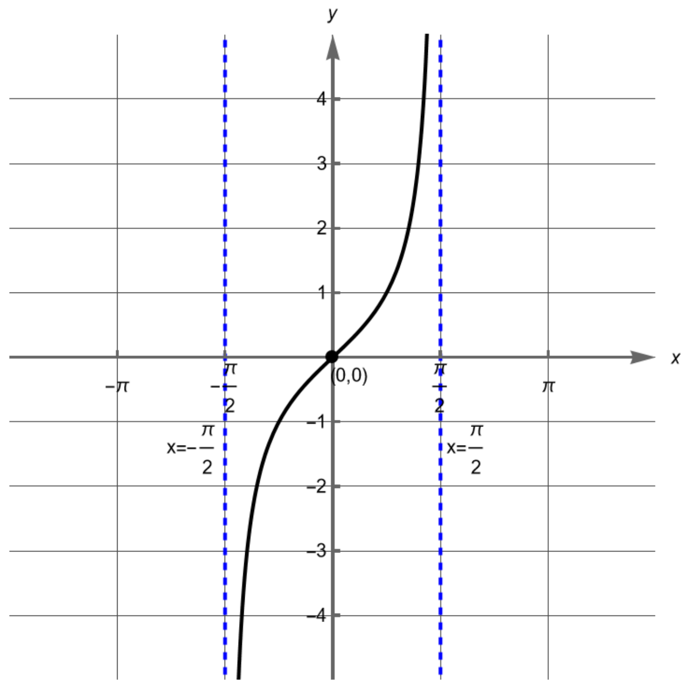
(b) $n(x) = \begin{cases} e^x & \text{if} \quad x \leq 0 \\ -2x+1 & \text{if} \quad 0 < x < 2 \\ \pi & \text{if} \quad x = 2 \end{cases}$

Solution:



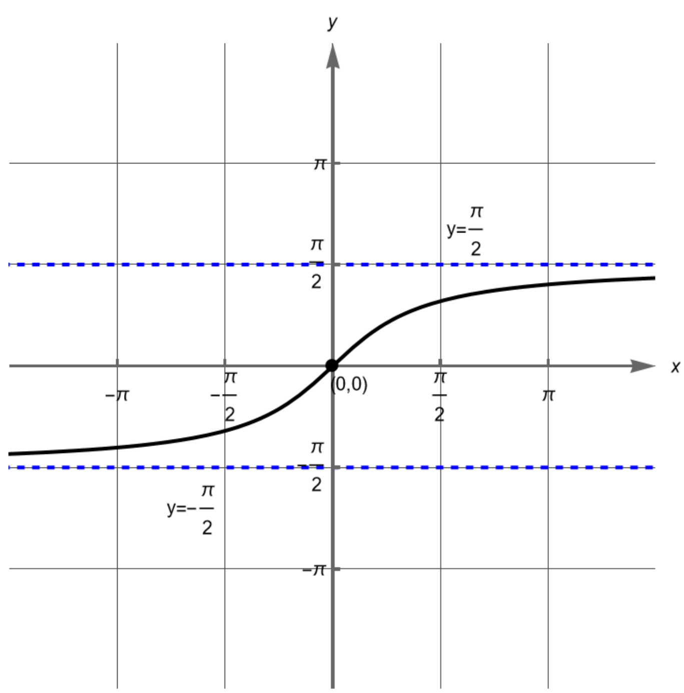
(c) $r(x) = \tan(x)$ on the restricted domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution:



(d) $j(x) = \tan^{-1}(x)$

Solution:



(e) For $j(x)$ in part (d) fill in the blank: $j(x) \to$ ____ as $x \to \infty$.

Solution:

$j(x) \to \boxed{\frac{\pi}{2}}$ as $x \to \infty$

16. Find the exact value for each (do not attempt to find decimal approximations): (12 pts)

(a) $\cos\left(\frac{\pi}{2}\right)$

Solution:

$\boxed{0}$

(b) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution:

$\boxed{\frac{\pi}{6}}$

(c) $\tan\left(-135^\circ\right)$

Solution:

$\boxed{1}$

(d) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

Solution:

$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \tag{61}$$

$$= \boxed{\frac{\pi}{3}} \tag{62}$$

17. Find the exact value for the given expression (do not attempt to find a decimal approximation): (4 pts)

$$\sin\left(-\frac{3\pi}{8}\right)$$

Solution:

First note that $-\frac{3\pi}{8}$ is in quadrant IV so $\sin\left(-\frac{3\pi}{8}\right)$ is negative. Now we apply the half angle formula:

$$\sin\left(-\frac{3\pi}{8}\right) = \sin\left(\frac{-\frac{3\pi}{4}}{2}\right) \tag{63}$$

$$= -\sqrt{\frac{1-\cos\left(-\frac{3\pi}{4}\right)}{2}} \tag{64}$$

$$= -\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} \tag{65}$$

$$= -\sqrt{\frac{2+\sqrt{2}}{4}} \tag{66}$$

$$= \boxed{-\frac{\sqrt{2+\sqrt{2}}}{2}} \tag{67}$$

18. Simplify the expression: $\dfrac{\cos(\theta)\sec(\theta)-\sin^2(\theta)}{2\cos(\theta)}$. (4 pts)

Solution:

$$\frac{\cos(\theta)\sec(\theta)-\sin^2(\theta)}{2\cos(\theta)} = \frac{\cos(\theta)\frac{1}{\cos(\theta)}-\sin^2(\theta)}{2\cos(\theta)} \tag{68}$$
$$= \frac{1-\sin^2(\theta)}{2\cos(\theta)} \tag{69}$$
$$= \frac{\cos^2(\theta)}{2\cos(\theta)} \tag{70}$$
$$= \boxed{\frac{\cos(\theta)}{2}} \tag{71}$$

19. Find all solutions to the following equations: (8 pts)

(a) $2\cos^2(\theta) = -\cos(\theta)$

Solution:

$$2\cos^2(\theta) = -\cos(\theta) \tag{72}$$
$$2\cos^2(\theta)+\cos(\theta) = 0 \tag{73}$$
$$\cos\theta(2\cos(\theta)+1) = 0 \tag{74}$$

By the multiplicative property of zero we set $\cos(\theta) = 0$ and $2\cos(\theta)+1 = 0$.

$\cos(\theta) = 0$ when $\theta = \frac{\pi}{2} + 2k\pi$ and $\theta = \frac{3\pi}{2} + 2k\pi$

$2\cos(\theta)+1 = 0$ when $\cos(\theta) = -\frac{1}{2}$. This happens when $\theta = \frac{2\pi}{3} + 2k\pi$ and $\theta = \frac{4\pi}{3} + 2k\pi$

Hence the solutions to the original equation are:

$\boxed{\theta = \dfrac{\pi}{2} + 2k\pi \text{ and } \theta = \dfrac{3\pi}{2} + 2k\pi \text{ and } \theta = \dfrac{2\pi}{3} + 2k\pi \text{ and } \theta = \dfrac{4\pi}{3} + 2k\pi, \text{ for } k \text{ any integer}}$

(b) $\cos(2x)\cos(x) + \sin(2x)\sin(x) = \dfrac{1}{2}$

Solution:

Using the difference formula for cosine from the formula sheet, we obtain:

$$\cos(2x)\cos(x)+\sin(2x)\sin(x) = \frac{1}{2} \tag{75}$$
$$\cos(2x-x) = \frac{1}{2} \tag{76}$$
$$\cos(x) = \frac{1}{2} \tag{77}$$

From which we conclude that $\boxed{x = \dfrac{\pi}{3} + 2k\pi \text{ and } x = \dfrac{5\pi}{3} + 2k\pi, \text{ for } k \text{ any integer}}$

20. For $m(x) = \frac{3}{2}\sin(4x)$ (7 pts)

(a) Identify the amplitude

Solution:

The amplitude is: $a = \left|\frac{3}{2}\right| = \boxed{\frac{3}{2}}$

(b) Identify the period.

Solution:

The period is: $\frac{\text{period of sine}}{|b|} = \frac{2\pi}{|4|} = \boxed{\frac{\pi}{2}}$

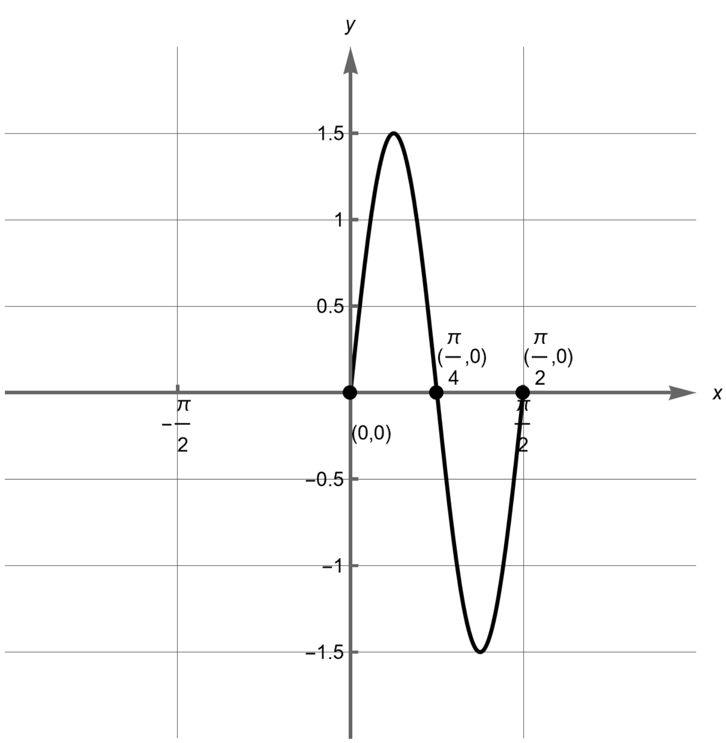
(c) Identify the phase shift

Solution:

The phase shift is $-\frac{c}{b} = -\frac{0}{4} = \boxed{0}$

(d) Sketch one cycle of the graph of $m(x)$. Label at least two values on the x-axis and clearly identify the amplitude.

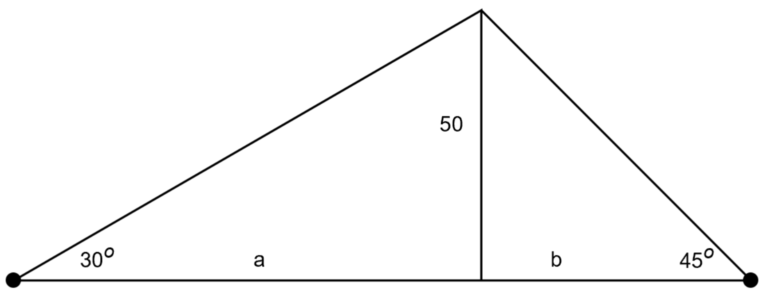
Solution:



21. Two surveyors are standing in a flat field with a 50 foot tall tree between them. The surveyors need to know how far apart they are standing from each other. One surveyor measures the angle of elevation from the ground where they stand to the top of the tree to be $30°$ while the other measures the angle of elevation from the ground where they stand to the top of the tree to be $45°$. How far apart, in feet, are the two surveyors? Give the exact answer, do not attempt to approximate with decimal values. (4 pts)

Solution:

In the following picture,the locations of the 2 surveyors are shown by bold dots



From the right triangle on the left, we can write:

$$\tan(30^o) = \frac{50}{a} \quad (78)$$

$$\frac{1}{\sqrt{3}} = \frac{50}{a} \quad (79)$$

$$a = 50\sqrt{3} \quad (80)$$

From the right triangle on the left, we can write:

$$\tan(45^o) = \frac{50}{b} \quad (81)$$

$$1 = \frac{50}{b} \quad (82)$$

$$b = 50 \quad (83)$$

Hence the distance between the 2 surveyors is given by:

$$d = a + b \quad (84)$$

$$= \boxed{50\sqrt{3} + 50 \text{ ft}} \quad (85)$$

$$= \boxed{50\left(1 + \sqrt{3}\right) \text{ ft}} \quad (86)$$