**APPM 3570/STAT 3100 — Exam 3 — Fall 2024**

On the front of your bluebook, write (1) **your name**, (2) **Exam 3**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers.** **Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.
Signature:____________________________ Date:_________

1. (30pts) There are 3 unrelated parts to this question. Justify your answers.

   (a) (10pts) Suppose we roll a pair of fair, six-sided dice. If $X$ is the number of dice that come up 1, and $Y$ is the number of dice that come up 4, 5 or 6, find $P(Y = 1|X = 0)$.

   (b) (10pts) Suppose $10$ balls are put into $5$ urns, with each ball independently being put in urn $i$ with probability $p_i$, where $\sum_{i=1}^{5} p_i = 1$. What is the expected number of urns that contains at least one ball?

   (c) (10pts) Suppose the random variables $X$, $Y$ and $Z$ have the means $\mu_X = 2$, $\mu_Y = -3$ and $\mu_Z = 4$, the variances $\sigma_X^2 = 1$, $\sigma_Y^2 = 5$ and $\sigma_Z^2 = 2$ and the covariances $\text{cov}(X, Y) = -2$, $\text{cov}(X, Z) = -1$ and $\text{cov}(Y, Z) = 1$. If we define $U = X - Y$ and $V = X + Z$, find $\text{cov}(U, V)$.

**Solution:** (a)(10pts) A table is helpful:

| Dice 1 \ Dice 2 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ |
| 2 | $(2,1)$ | $(2,2)$ | $(2,3)$ | $(2,4)$ | $(2,5)$ | $(2,6)$ |
| 3 | $(3,1)$ | $(3,2)$ | $(3,3)$ | $(3,4)$ | $(3,5)$ | $(3,6)$ |
| 4 | $(4,1)$ | $(4,2)$ | $(4,3)$ | $(4,4)$ | $(4,5)$ | $(4,6)$ |
| 5 | $(5,1)$ | $(5,2)$ | $(5,3)$ | $(5,4)$ | $(5,5)$ | $(5,6)$ |
| 6 | $(6,1)$ | $(6,2)$ | $(6,3)$ | $(6,4)$ | $(6,5)$ | $(6,6)$ |

| Dice 1 \ Dice 2 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ |
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| 3 | $(3,1)$ | $(3,2)$ | $(3,3)$ | $(3,4)$ | $(3,5)$ | $(3,6)$ |
| 4 | $(4,1)$ | $(4,2)$ | $(4,3)$ | $(4,4)$ | $(4,5)$ | $(4,6)$ |
| 5 | $(5,1)$ | $(5,2)$ | $(5,3)$ | $(5,4)$ | $(5,5)$ | $(5,6)$ |
| 6 | $(6,1)$ | $(6,2)$ | $(6,3)$ | $(6,4)$ | $(6,5)$ | $(6,6)$ |

The range of the rv's is $X, Y \in \{0, 1, 2\}$ and, by definition

$$P(Y = 1|X = 0) = \frac{P(X = 0, Y = 1)}{P(X = 0)}$$

where $\{X = 0\}^c = \{X \neq 0\} = \{(1,1),\, (1,i),\, (i,1)\,|\, 2 \leq i \leq 6\}$, so

$$P(X = 0) = 1 - P(X \neq 0) = 1 - \frac{11}{36} = \frac{25}{36}$$

and $\{X = 0, Y = 1\} = \{(2,i),\, (3,i),\, (i,2),\, (i,3)\,|\, 4 \leq i \leq 6\}$, so $P(X = 0, Y = 1) = \dfrac{12}{36}$, thus,

$$P(Y = 1|X = 0) = \frac{P(X = 0, Y = 1)}{P(X = 0)} = \frac{12/36}{25/36} = \frac{12}{25}.$$

(b)(10pts) For $1 \leq i \leq 5$, let

$$X_i = \begin{cases} 1, & \text{if there is at least } 1 \text{ ball in urn } i, \\ 0, & \text{otherwise.} \end{cases}$$

then $P(X_i = 1) = 1 - P(X_i = 0) = 1 - (1 - p_i)^{10}$. Now let $X = \sum_{i=1}^{5} X_i$, then the expected number of urns that contain at least one ball is

$$E(X) = E\Big(\sum_{i=1}^{5} X_i\Big) = \sum_{i=1}^{5} E(X_i) = \sum_{i=1}^{5} P(X_i = 1) = \sum_{i=1}^{5} [1 - (1 - p_i)^{10}].$$

(c)(10pts) Note that, if $U = X - Y$ and $V = X + Z$, then

$$\begin{aligned}\text{cov}(U, V) = \text{cov}(X - Y, X + Z) &= \text{cov}(X - Y, X) + \text{cov}(X - Y, Z) \\ &= \text{cov}(X, X) + \text{cov}(-Y, X) + \text{cov}(X, Z) + \text{cov}(-Y, Z) \\ &= \text{cov}(X, X) - \text{cov}(X, Y) + \text{cov}(X, Z) - \text{cov}(Y, Z) = 1 - (-2) + (-1) - 1 = 1.\end{aligned}$$

2. (40pts) A nut company markets cans of deluxe mixed nuts containing *almonds*, *cashews*, and *peanuts*. Suppose the net weight of each can is $1$ lb, but the weight contribution of each type of nut is random. Let $X$ be the weight of almonds in a selected can and $Y$ the weight of cashews. The joint pdf of $(X, Y)$ is given to be

$$f(x, y) = \begin{cases} 24xy, & \text{for } 0 < x < 1,\ 0 < y < 1 \text{ and } x + y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

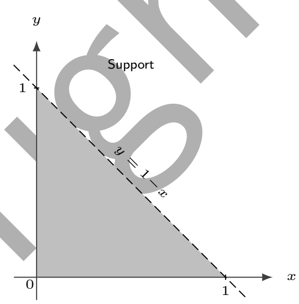
(a) (10pts) Set up, but *do not solve*, an integral (or integrals) to find the probability that the *peanuts* make up at least $60\%$ of the can.

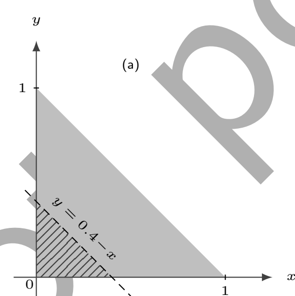
(b) (10pts) Find $f_X(X)$, the marginal pdf for the weight of almonds, $X$, and find the expectation $E[X]$. (Be sure to define the pdf for all values of $\mathbb{R}$.)

(c) (10pts) Suppose $1$ lb of almonds fills up $2.8$ cups, $1$ lb of cashews fills $3.5$ cups and $1$ lb of peanuts fills $2.7$ cups. Find the expected volume of the contents of a randomly selected can of deluxe nuts (in terms of cups). (*Hint:* No further calculations are needed to find $E[Y]$.)

(d) (10pts) Find the conditional pdf $Y$ given that $X = 0.75$ and the conditional expectation of $Y|X = 0.75$.

**Solution:**




(a)(10pts) If $Z$ is the weight of the peanuts (in lbs), then $Z = 1 - X - Y$, and so

$$P(Z \geq 0.6) = P(1 - X - Y \geq 0.6) = P(X + Y \leq 0.4) = P(0 < X < 0.4,\ 0 \leq Y \leq 0.4 - X) = \int_0^{0.4} \int_0^{0.4 - x} 24xy\, dy dx.$$

(b)(10pts) First note that, for each $x \in (0, 1)$, we have

$$f_X(x) = \int_{\mathbb{R}} f(x, y)\, dy = \int_0^{1-x} 24xy\, dy = 24x \cdot \frac{y^2}{2}\bigg|_0^{1-x} = 12x(1-x)^2 \text{ for } x \in (0,1) \text{ and } 0 \text{ otherwise.}$$

Now, using the marginal density function $f_X(x)$, we have

$$\begin{aligned}E[X] = \int_{\mathbb{R}} x f_X(x) dx = \int_0^1 x\left[12x(1-x)^2\right] dx &= 12 \int_0^1 x^2(1 - 2x + x^2) dx \\ &= 12 \int_0^1 (x^2 - 2x^3 + x^4)\, dx \\ &= 12\left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right)\bigg|_0^1 = 12\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{12}{30} = \frac{2}{5}.\end{aligned}$$

(c)(10pts) Let $V$ be the volume of the contents of a can of deluxe nuts. Since $X$, $Y$ and $Z$ are the weight in pounds of almonds, cashews and peanuts respectfully, we have $V = 2.8X + 3.5Y + 2.7Z$ and, since $Z = 1 - X - Y$, we have

$$V = 2.8X + 3.5Y + 2.7Z = 2.8X + 3.5Y + 2.7(1 - X - Y) = 2.7 + 0.1X + 0.8Y$$

Now note that, by symmetry, we have

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dy = \int_0^1 x(12x - 24x^2 + 12x^3)\,dx \;\Rightarrow\; E[X] = E[Y] = \frac{2}{5}$$

thus, the expected volume of the contents is

$$\begin{aligned} E[V] = E[2.7 + 0.1X + 0.8Y] &= 2.7 + \frac{1}{10}E[X] + \frac{8}{10}E[Y] \\ &= \frac{27}{10} + \frac{1}{10}\cdot\frac{2}{5} + \frac{8}{10}\cdot\frac{2}{5} = \frac{135}{50} + \frac{18}{50} = \frac{153}{50} \;\Rightarrow\; E[V] = 3.06 \text{ cups.} \end{aligned}$$

(d)(5pts)$(i)$ Note that $f_X(0.75) = 12\cdot\frac{3}{4}\cdot\frac{1}{16} = \frac{9}{16}$ and, for each $x \in (0,1)$, we have $0 < y < 1 - x$, thus, the conditional pdf $Y|X=0.75$ is

$$f_{Y|X}(y|0.75) = \frac{f(x,0.75)}{f_X(0.75)} = \begin{cases} \dfrac{18y}{9/16}, & \text{for } 0 < y < \frac{1}{4}, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 32y, & \text{for } 0 < y < \frac{1}{4}, \\ 0, & \text{otherwise.} \end{cases}$$

(d)(5pts)$(ii)$ The conditional expectation of $Y|X=0.75$ is

$$\begin{aligned} E[Y|X = 0.75] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dy &= \int_0^{1/4} y \cdot 32y\,dy \\ &= \int_0^{1/4} 32y^2\,dy = 32\cdot\frac{y^3}{3}\Big|_0^{1/4} = 32\cdot\frac{1}{3}\cdot\frac{1}{64} = \frac{1}{6}. \end{aligned}$$

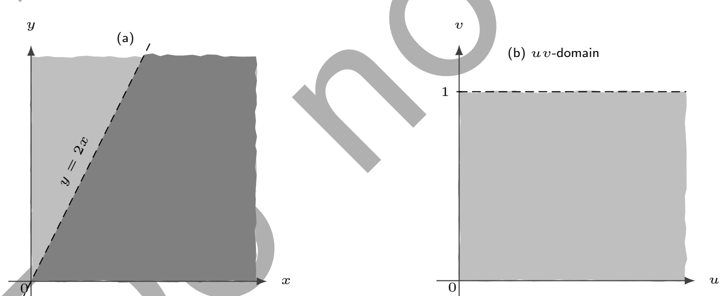
3. (30pts) Let $X$ and $Y$ be independent and identically distributed *Exponential* random variables each with parameter $\lambda = 1$.

   (a) (10pts) Find the probability $P\Big(\dfrac{Y}{X} < 2\Big)$. Show all work.

   (b) (10pts) Let $U = X + Y$ and $V = \dfrac{X}{X+Y}$, find $f_{U,V}(u,v)$, the *joint probability density function* of $(U,V)$.

   (c) (10pts) Find the marginal pdf of the random variable $V$, be sure to specify the domain. Are $U$ and $V$ independent? Why or why not?

**Solution:**




(a)(10pts) First, since $X$ and $Y$ are iid $\text{Exponential}(\lambda = 1)$, the joint pdf of $(X,Y)$ is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} e^{-x}e^{-y}, & \text{for } x > 0, y > 0, \\ 0, & \text{elsewhere,} \end{cases} = \begin{cases} e^{-(x+y)}, & \text{for } x > 0, y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Thus, we have (see graph)

$$\begin{aligned} P\Big(\frac{Y}{X} < 2\Big) &= P\big(Y < 2X\big) \\ &= P\big(0 < X < \infty,\, 0 < Y < 2X\big) \\ &= \int_0^{\infty}\int_0^{2x} e^{-(x+y)}\,dydx \\ &= \int_0^{\infty} -e^{-(x+y)}\Big|_0^{2x}\,dx = \int_0^{\infty}(-e^{-3x} + e^{-x})\,dx = \Big(\frac{e^{-3x}}{3} - e^{-x}\Big)\Big|_0^{\infty} = 0 - \Big(\frac{1}{3} - 1\Big) = \frac{2}{3}. \end{aligned}$$

Note, we also have $P\big(\frac{Y}{X} < 2\big) = P\big(Y < 2X\big) = P(\frac{Y}{2} < X < \infty, 0 < Y < \infty)$.

(b)(10pts) Solving for $(X, Y)$ in terms of $(U, V)$ gives

$$U = X + Y \;\Rightarrow\; V = \frac{X}{X+Y} = \frac{X}{U} \;\Rightarrow\; \begin{cases} X = UV, \\ Y = U - X = U - UV = U(1-V). \end{cases}$$

For the domain, note that $x > 0$ and $y > 0$ implies $u = x + y > 0$ and $v = \frac{x}{x+y} > 0$, thus,

$$x > 0 \;\Rightarrow\; uv > 0 \;\Rightarrow\; u > 0,\, v > 0 \text{ and } y > 0 \;\Rightarrow\; u - uv > 0 \;\Rightarrow\; u > uv \;\Rightarrow\; 1 > v.$$

Finally, recall that $f_{U,V}(u, v) = f_{X,Y}\big(x(u, v), y(u, v)\big)|J(u, v)|$, where

$$J(u,v) = \begin{vmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv - u(1-v) = -uv - u + uv = -u,$$

thus, we have

$$f_{U,V}(u,v) = f_{X,Y}\big(x(u,v), y(u,v)\big)|J(u,v)| = f_{X,Y}\big(uv, u - uv\big) \cdot u$$
$$= \begin{cases} e^{-(uv+u-uv)} \cdot u, & \text{for } 0 < u,\, 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases} = \begin{cases} ue^{-u}, & \text{for } 0 < u,\, 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(c)(5pts)$(i)$ Using integration by parts (or the fact that $g(t) = te^{-t}, t > 0$ is the pdf of Gamma rv), the marginal probability density function of $V$ is given by

$$f_V(v) = \int_{-\infty}^{\infty} f(u,v)\,du = \int_0^{\infty} ue^{-u}\,du = -ue^{-u}\Big|_0^{\infty} + \int_0^{\infty} e^{-u}\,du = -e^{-u}\Big|_0^{\infty} = 0 - (-1) = 1 \text{ for } 0 < v < 1 \text{ and } 0 \text{ otherwise.}$$

(c)(5pts)$(ii)$ Yes, they are independent. Note that $\frac{f(u,v)}{f_V(v)}$ is strictly a function of $u$. Just for kicks, note that the marginal probability density function of $U = X + Y$ is given by

$$f_U(u) = \int_{-\infty}^{\infty} f(u,v)\,dv = \int_0^1 ue^{-u}\,dv = ue^{-u} \text{ for } u > 0 \text{ and } 0 \text{ otherwise.}$$