1. (34 points) The following problems are unrelated.

   (a) Find the tangent line of $f(x) = \sqrt{x^5} + \dfrac{4x^8}{3ax^{2/3}}$ at $x = 1$. Write your final answer in point-slope form. Your answer will be in terms of the constant a.

   (b) Evaluate $\displaystyle\int_1^e \frac{1}{x(\ln x)^2 + x}\, dx$.

   (c) Evaluate $\displaystyle\int \frac{\sinh(\tan x + 4)}{\cos^2 x}\, dx$.

   Solution:

   (a) **Similar to Exam 2, Problem 1(a) and Exam 3, Problem 1(a)**

   Note that $f(x) = x^{5/2} + \dfrac{4}{3a}x^{22/3}$. So, we have $f'(x) = \frac{5}{2}x^{3/2} + \frac{88}{9a}x^{19/3}$. We have $f(1) = 1 + \frac{4}{3a}$ and $f'(1) = \frac{5}{2} + \frac{88}{9a}$. So, the tangent line is

$$y - \left(1 + \frac{4}{3a}\right) = \left(\frac{5}{2} + \frac{88}{9a}\right)(x - 1).$$

   (b) We will apply the substitution $u = \ln x$. This yields $du = \frac{1}{x}dx$, new upper limit of integration $u = \ln(e) = 1$, and new lower limit of integration $u = \ln(1) = 0$:

$$\begin{aligned}\int_1^e \frac{1}{x(\ln x)^2 + x}\, dx &= \int_0^1 \frac{1}{u^2+1} du\\ &= \arctan(1) - \arctan(0)\\ &= \frac{\pi}{4}.\end{aligned}$$

   (c) We will apply the substitution $u = \tan x + 4$. This yields $du = \sec^2 x\, dx = \dfrac{dx}{\cos^2 x}$:

$$\begin{aligned}\int \frac{\sinh(\tan x + 4)}{\cos^2 x}\, dx &= \int \sinh(u)\, du\\ &= \cosh(u) + C\\ &= \cosh(\tan x + 4) + C.\end{aligned}$$

2. (18 points) Consider $g(x) = x^{1/(1-x)}$ as you answer both of the following.

   (a) Evaluate $\displaystyle\lim_{x\to 1^+} g(x)$.

   (b) Find $g'(x)$. (Find $g'(x)$ in terms of x, but please DO NOT simplify your answer further.)

   Solution:

(a) Since the given limit is a 1^∞ indeterminate power, then we will first apply the natural logarithm to the limit:

$$\begin{aligned} L &= \ln\left(\lim_{x\to1^+} x^{1/(1-x)}\right) \\ &= \lim_{x\to1^+} \frac{\ln x}{1-x}. \end{aligned}$$

We see that this limit is a $\frac{0}{0}$-indeterminate form, so we can apply L'Hospital's rule. This yields

$$\begin{aligned} L &= \lim_{x\to1^+} \frac{1/x}{-1} \\ &= -1. \end{aligned}$$

So,

$$\lim_{x\to1^+} g(x) = e^L = \frac{1}{e}.$$

(b) First, we apply the natural logarithm to both sides and we expand:

$$\ln(g(x)) = \ln\left(x^{1/(1-x)}\right) = \frac{\ln x}{1-x}.$$

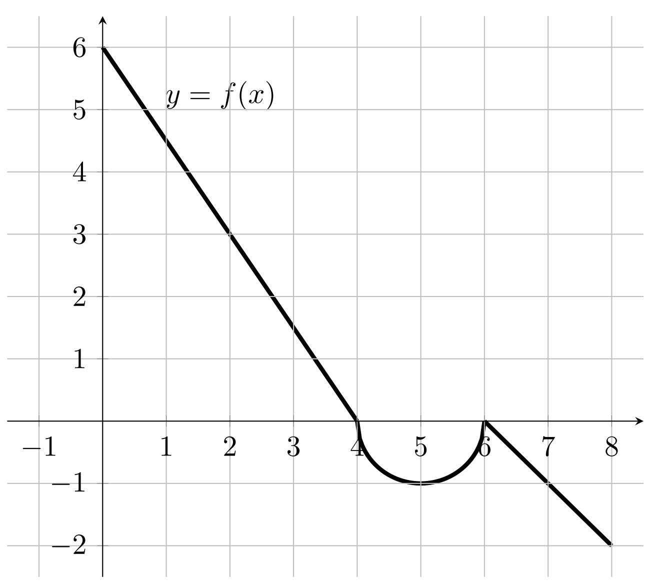
Next, we differentiate both sides to obtain

$$\frac{g'(x)}{g(x)} = \frac{(1-x)\cdot(1/x) + \ln x}{(1-x)^2} = \frac{1/x - 1 + \ln x}{(1-x)^2}.$$

Lastly, we solve for $g'(x)$ and write it in terms of x to obtain

$$g'(x) = x^{1/(1-x)}\left(\frac{(1-x)\cdot(1/x) + \ln x}{(1-x)^2}\right) = x^{1/(1-x)}\left(\frac{1/x - 1 + \ln x}{(1-x)^2}\right).$$

3. (22 points) Consider the function $f(x)$ defined over $[0, 8]$ that is graphed below. It consists of two straight line segments and a semicircle.



(a) Find the average value of f over the interval $[0, 8]$.

(b) Evaluate $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.

(c) Let $g(x) = \int_0^x f(t)\, dt$. Find the tangent line of $y = g(x)$ at $x = 2$.

Solution:

(a) By inspecting the graph, we obtain:

$$f_{avg} = \frac{1}{8} \int_0^8 f(x)\, dx = \frac{1}{8} \left(12 - \frac{\pi}{2} - 2\right) = \frac{20 - \pi}{16}.$$

(b) Note that the slope at $x = 3$ is $-3/2$ and $f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$ so

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \frac{-3}{2}.$$

(c) We see that

$$g(2) = \int_0^2 f(t)\, dt = 9.$$

We also see

$$g'(2) = f(2) = 3.$$

So, the tangent line is

$$y = g(2) + g'(2)(x - 2)$$
$$y = 9 + 3(x - 2)$$
$$y = 3x + 3.$$

4. (21 points) The following two problems are not related.

(a) Particle A moves along an axis. Its velocity on this axis is provided for specific times in the table below.

time (seconds)	0	2	4	6	8	10	12
velocity (meters per second)	10	12	8	5	0	2	5

Approximate the displacement of Particle A from $t = 0$ to $t = 12$ in two ways:

i. Using the upper sum rule with $n = 6$ intervals of equal width.

ii. Using the midpoint rule with $n = 3$ intervals of equal width.

(b) Suppose Particle B moves along a different axis, with an initial velocity of 20 m/s. If it comes to a complete stop after traveling 60 meters, what must its constant rate of acceleration have been?

Solution:

(a) i.

$$2(12 + 12 + 8 + 5 + 2 + 5) = 88 \text{ meters}.$$

ii.

$$4(12 + 5 + 2) = 76 \text{ meters}.$$

(b) **Similar to Homework #9, Problem 8**

We will call this constant acceleration C. So, we have $a(t) = C$. It follows from this and the given information that $v(t) = Ct + 20$ and $s(t) = \frac{1}{2}Ct^2 + 20t$, if we assume the initial position is 0. (If we chose a different initial position, it would not change our answer.)

We know that the velocity will be 0 meters per second when Particle B stops. Solving $v(t) = 0$ yields $t = -\frac{20}{C}$. Plugging this into the position function we have

$$\begin{aligned} 60 &= s\left(-\frac{20}{C}\right) \\ &= \frac{1}{2}C\left(-\frac{20}{C}\right)^2 + 20\left(-\frac{20}{C}\right) \\ &= -\frac{200}{C}. \end{aligned}$$

Thus, the constant acceleration is $C = -\frac{10}{3}$ meters per square second.

5. (24 pts) Consider $r(x) = \dfrac{e^{2x}}{e^{2x}+3}$.

(a) Determine the horizontal asymptote(s) of $y = r(x)$, or show that there are none. Be sure to justify your answer with the appropriate limits.

(b) Find $r'(x)$ and use this to show that r is one-to-one.

(c) Determine the formula for the inverse function, $r^{-1}(x)$. (Clearly label your final answer as $r^{-1}(x)$.)

(d) Determine the domains and ranges of $r(x)$ and $r^{-1}(x)$.

Solution: For some of these problems, it is helpful to note that $r(x) = \dfrac{e^{2x}}{e^{2x}+3} \cdot \dfrac{e^{-2x}}{e^{-2x}} = \dfrac{1}{1+3e^{-2x}}$.

(a) We check both limits at infinity:

$$\lim_{x\to-\infty} r(x) = \lim_{x\to-\infty} \frac{1}{1+3e^{-2x}} = 0$$

and

$$\lim_{x\to\infty} r(x) = \lim_{x\to\infty} \frac{1}{1+3e^{-2x}} = 1.$$

So, $y = r(x)$ has horizontal asymptotes of $y = 0, 1$.

(b) Using our simplified form of $r(x)$, we differentiate to find $r'(x) = \dfrac{6e^{-2x}}{(1+3e^{-2x})^2}$, which is always positive. Since $r(x)$ is always increasing, we see that it is one-to-one.

(c) We switch x and y, and then solve for y:

$$\begin{aligned} x &= \frac{1}{1+3e^{-2y}} \\ 1+3e^{-2y} &= \frac{1}{x} \\ e^{-2y} &= \frac{1-x}{3x} \end{aligned}$$

$$-2y = \ln\left(\frac{1-x}{3x}\right)$$
$$y = -\frac{1}{2}\ln\left(\frac{1-x}{3x}\right)$$
$$r^{-1}(x) = \ln\sqrt{\frac{3x}{1-x}}.$$

(d) Note that $r(x)$ is defined for all real numbers. Also, we see that its range is $(0,1)$ given the value of the limits we found in (a) and the fact that $r'(x) > 0$ for all x. So, we have

$$\text{Domain of } r(x) = \text{Range of } r^{-1}(x) = (-\infty, \infty)$$

$$\text{Range of } r(x) = \text{Domain of } r^{-1}(x) = (0,1)$$

6. (15 points) At 12pm, the population of a bacteria culture is 5,000. By 3pm, the population is 6,000. Assuming the growth of the population is proportional to the current size of the population, that is $\frac{dP}{dt} = kP$, what will the population be at 7pm? (We know you do not have a calculator. Leave your answer as an exact answer.)

Solution: We will model the population with $P(t) = P_0 e^{kt}$ where t measures hours since 12pm and $P(t)$ measures thousands of bacteria. We have $P(0) = 5$, $P(3) = 6$, and we want to know $P(7)$. Note that $P_0 = 5$. We can find k with the second data point:

$$P(3) = 6$$
$$5e^{3k} = 6$$
$$3k = \ln\frac{6}{5}$$
$$k = \frac{1}{3}\ln\frac{6}{5}.$$

So, the population at 7pm is expected to be $P(7) = 5e^{\frac{7}{3}\ln\frac{6}{5}}$ thousand.

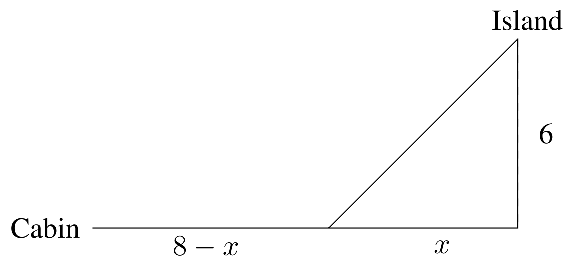
7. (16 points) An island is 6 miles due north of its closest point along a straight shoreline. A cabin on the shoreline is 8 miles west of that point. Jane is currently on the island and is planning to go from the island to the cabin. Jane has access to a sailboat. Jane can run along the shoreline at a rate of 5 mph and sail through the water at a rate of 3 mph. Jane is not allowed to go east of the island or west of the cabin, and she can only sail in a straight line. Let x represent the number of miles down the shoreline Jane will sail before running the rest of the way.

Note: $15^2 = 225$.

(a) Give the formula for a function for the total time it takes to go from the island to the cabin. This function should be in terms of x.

(b) How far down the shoreline should Jane sail before running the rest of the way to **minimize** the time it takes to reach the cabin? (Be sure to justify that you have found the absolute minimum.)

(c) How far down the shoreline should Jane sail before running the rest of the way to **maximize** the time it takes to reach the cabin? (Assume Jane must run and sail at the rates mentioned above. Be sure to justify that you have found the absolute maximum.)



Solution:

Similar to Homework #9, Problem 3

Recall that for a fixed speed, we have

$$(\text{distance}) = (\text{speed})(\text{time}).$$

We use this to construct our time function.

We have the time spent running as $\frac{8-x}{5}$ and the time spend sailing as $\frac{\sqrt{x^2+36}}{3}$, where the numerator was found with the Pythagorean theorem.

(a) The time function is

$$T(x) = \frac{8-x}{5} + \frac{\sqrt{x^2+36}}{3}$$

So, we want to find the absolute maximum and minimum of $T(x)$ for x in $[0, 8]$. We will test this function at the endpoints and any critical numbers (using the Extreme Value Theorem) to determine these absolute extrema.

We see that

$$T'(x) = -\frac{1}{5} + \frac{x}{3\sqrt{x^2+36}},$$

which exists on $[0, 8]$. We will solve $T'(x) = 0$:

$$\begin{aligned}
\frac{x}{3\sqrt{x^2+36}} &= \frac{1}{5} \\
5x &= 3\sqrt{x^2+36} \\
25x^2 &= 9(x^2+36) \\
x^2 &= \frac{9 \cdot 36}{16} \\
x &= \pm\frac{3 \cdot 6}{4} = \pm\frac{9}{2}.
\end{aligned}$$

Because of our domain, $x = 9/2$ is our only critical number. Then, we note that

$$T(0) = \frac{8}{5} + \frac{6}{3} = \frac{18}{5} = 3.6,$$

$$T(9/2) = \frac{1}{5} \cdot \frac{7}{2} + \frac{1}{3}\sqrt{81/4 + 36} = \frac{7}{10} + \frac{\sqrt{225}}{6} = \frac{16}{5} = 3.2,$$

and

$$T(8) = 0 + \frac{10}{3} = 3.\overline{3}.$$

(b) To minimize travel time, Jane should sail to a point 9/2 miles down the shoreline.

(c) To maximize travel time, Jane should sail to a point 0 miles down the shoreline.