1. (20 pts) The position function of a particle is given by $s(t) = t^{3/2} - 3t + 10$ on the interval $1 \le t \le 9$, where position is in meters and time is in seconds.

   (a) Determine the particle's velocity function $v(t)$. Include the correct unit of measurement.

   Solution:

   $$v(t) = s'(t) = \boxed{\frac{3}{2}t^{1/2} - 3 = \frac{3}{2}\sqrt{t} - 3 \text{ m/s}}\,, \quad 1 < t < 9$$

   (b) Determine the total distance traveled by the particle on the interval $1 \le t \le 9$. Include the correct unit of measurement.

   Solution:

   $$v(t) = \frac{3}{2}\sqrt{t} - 3 = 0 \quad \Rightarrow \quad \frac{\sqrt{t}}{2} = 1 \quad \Rightarrow \quad \sqrt{t} = 2 \quad \Rightarrow \quad t = 4$$

   Since $v(t)$ changes sign at $t = 4$, we need to calculate the distances traveled during the time intervals $[1, 4]$ and $[4, 9]$ separately, and add those results together.

   Distance traveled on time interval $[1, 4] = |s(4) - s(1)|$

   Distance traveled on time interval $[4, 9] = |s(9) - s(4)|$

   $s(1) = 1^{3/2} - (3)(1) + 10 = 1 - 3 + 10 = 8$

   $s(4) = 4^{3/2} - (3)(4) + 10 = 8 - 12 + 10 = 6$

   $s(9) = 9^{3/2} - (3)(9) + 10 = 27 - 27 + 10 = 10$

   Distance traveled on time interval $[1, 4] = |s(4) - s(1)| = |6 - 8| = 2$

   Distance traveled on time interval $[4, 9] = |s(9) - s(4)| = |10 - 6| = 4$

   Therefore, the total distance traveled between $t = 1$ and $t = 9$ seconds is $2 + 4 = \boxed{6 \text{ m}}$

2. (17 pts) Consider the function $f(x) = \dfrac{(x-2)^2}{x-5}$.

(a) Identify all critical numbers of f on the interval $(-\infty, \infty)$.

Solution:

Critical numbers are values of x in the domain of f such that $f'(x) = 0$ or $f'(x)$ is undefined.

$$f'(x) = \frac{(x-5)\,[2(x-2)] - (x-2)^2}{(x-5)^2} = \frac{(x-2)\,[(2x-10)-(x-2)]}{(x-5)^2} = \frac{(x-2)(x-8)}{(x-5)^2}$$

Since $f'(2) = 0$ and $f'(8) = 0$, $\boxed{x = 2, 8}$ are critical numbers of $f(x)$.

Note that although $f'(5)$ is undefined, $x = 5$ is not a critical number of $f(x)$ because $x = 5$ is not in the domain of $f(x)$.

(b) The Extreme Value Theorem indicates that $f(x)$ attains an absolute maximum value and an absolute minimum value on the interval $[0, 4]$. Identify the (x, y) coordinates of every point at which f attains one of those extreme values on that interval. Be sure to clearly indicate which point(s) correspond to the maximum value and which correspond to the minimum value.

(*Hint:* Use the Closed Interval Method.)

Solution:

The Extreme Value Theorem applies to $f(x)$ on $[0, 4]$ because $[0, 4]$ is a **closed** interval and f is **continuous** on that interval. Therefore, the Closed Interval Method can be applied.

The Closed Interval Method involves evaluating the function at all critical numbers on the interval $(0, 4)$ and evaluating the function at the interval boundaries. The only critical number on $(0, 4)$ is $x = 2$ and the boundaries correspond to $x = 0$ and $x = 4$.

$$f(0) = \frac{(0-2)^2}{0-5} = -4/5$$

$$f(2) = \frac{(2-2)^2}{2-5} = 0$$

$$f(4) = \frac{(4-2)^2}{4-5} = -4$$

Since $-4 < -4/5 < 0$, the absolute maximum value of f occurs at the point $\boxed{(2, 0)}$ and the absolute minimum value of f occurs at the point $\boxed{(4, -4)}$

3. (19 pts) Find an equation of the tangent line to the curve $\sin(x-y) = \sin y - \cos x$ at the point $(\pi/4, \pi/4)$.

Solution:

We begin by executing implicit differentiation, as follows:

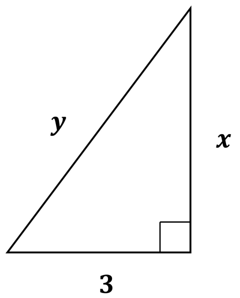
$$\begin{aligned}
\frac{d}{dx}[\sin(x-y)] &= \frac{d}{dx}[\sin y - \cos x] \\
\cos(x-y)\cdot(1-y') &= \cos y \cdot y' + \sin x \\
[\cos(x-y) + \cos y]\, y' &= \cos(x-y) - \sin x \\
y' &= \frac{\cos(x-y) - \sin x}{\cos(x-y) + \cos y}
\end{aligned}$$

The slope of the tangent line is calculated next.

$$\begin{aligned}
y'\Big|_{x=y=\pi/4} &= \frac{\cos(\pi/4 - \pi/4) - \sin(\pi/4)}{\cos(\pi/4 - \pi/4) + \cos(\pi/4)} \\
&= \frac{\cos(0) - \sin(\pi/4)}{\cos(0) + \cos(\pi/4)} \\
= &= \frac{1 - 1/\sqrt{2}}{1 + 1/\sqrt{2}} \\
= &= \frac{1 - 1/\sqrt{2}}{1 + 1/\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2} - 1}{\sqrt{2} + 1}
\end{aligned}$$

Therefore, point-slope form of the tangent line is $\boxed{y - \pi/4 = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)(x - \pi/4)}$

4. (16 pts) The following questions relate to the right triangle depicted here:



(a) Write an expression for y as a function of x.

Solution:

$\boxed{y(x) = \sqrt{9+x^2}}$, per the Pythagorean Theorem.

(b) Find the linear approximation of $y(x)$ at $a = 4$.

Solution:

$$y(x) \approx L(x) = y(4) + y'(4)(x-4)\ , \quad x \approx 4$$

$$y(4) = \sqrt{9+4^2} = 5$$

$$y'(x) = \frac{1}{2}\,(9+x^2)^{-1/2}(2x) = \frac{x}{\sqrt{9+x^2}} \quad \Rightarrow \quad y'(4) = \frac{4}{\sqrt{9+4^2}} = \frac{4}{5}$$

$$L(x) = \boxed{5 + \frac{4}{5}(x-4)}$$

(c) Suppose the value of x is measured to be 4 cm with a possible error in measurement of at most 0.25 cm. Use differentials to estimate the maximum error in computing the value of y based on the measured value of x. Include the correct unit of measurement.

Solution:

$$dx = 0.25$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{9+x^2}} \quad \Rightarrow \quad dy = \frac{x}{\sqrt{9+x^2}} \cdot dx$$

$$dy\Big|_{x=4} = \frac{4}{\sqrt{9+4^2}} \cdot (0.25) = \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5} = \boxed{0.2 \text{ cm}}$$

5. (20 pts) Parts (a) and (b) are not related.

(a) At $x = 2$, does the function $g(x) = \dfrac{x^2 - 5x + 6}{x^2 - 4x + 4}$ have a removable discontinuity, a vertical asymptote, or neither? Support your answer by evaluating the appropriate limit(s).

Solution:

$$g(x) = \frac{x^2 - 5x + 6}{x^2 - 4x + 4} = \frac{(x-2)(x-3)}{(x-2)^2} = \frac{x-3}{x-2}, \quad x \neq 2$$

$$\lim_{x\to 2^-} \frac{x-3}{x-2} \to \frac{2-3}{0^-} \to \frac{-1}{0^-} \to \infty$$

$$\lim_{x\to 2^+} \frac{x-3}{x-2} \to \frac{2-3}{0^+} \to \frac{-1}{0^+} \to -\infty$$

Either of the preceding two infinite limits is sufficient to establish that $g(x)$ has a $\boxed{\text{vertical asymptote at } x = 2}$

(b) Find the equation of each horizontal asymptote of $y = h(x) = \dfrac{2x - 3x\sqrt{x} + 1}{2x^{3/2} + 3x - 1}$, if any exist. Support your answer by evaluating the appropriate limit(s).

(Note: You may not use L'Hôpital's Rule or dominance of powers arguments to evaluate limits on this exam.)

Solution:

$$\begin{aligned} h(x) &= \frac{2x - 3x\sqrt{x} + 1}{2x^{3/2} + 3x - 1} = \frac{2x - 3x^{3/2} + 1}{2x^{3/2} + 3x - 1} \\ &= \frac{2x - 3x^{3/2} + 1}{2x^{3/2} + 3x - 1} \cdot \frac{x^{-3/2}}{x^{-3/2}} \\ &= \frac{2x^{-1/2} - 3 + x^{-3/2}}{2 + 3x^{-1/2} - x^{-3/2}} \end{aligned}$$

$$\lim_{x\to\pm\infty} \frac{2x^{-1/2} - 3 + x^{-3/2}}{2 + 3x^{-1/2} - x^{-3/2}} = \frac{0 - 3 + 0}{2 + 0 - 0} = -\frac{3}{2}$$

The preceding limit indicates that $h(x)$ has a $\boxed{\text{horizontal asymptote at } y = -3/2}$

6. (18 pts) Verify that the hypotheses of the Mean Value Theorem are satisfied for the function $u(x) = x - \cos^3 x$ on the interval $[0, 2\pi]$, and find all numbers c that satisfy the conclusion of that theorem.

Solution:

Since polynomials and the cosine function are continuous and differentiable on the interval $(-\infty, \infty)$ and $u(x)$ is the sum of two such functions, the hypotheses of the Mean Value Theorem are both satisfied:

$\boxed{u(x) \text{ is continuous on } [0, 2\pi]}$ and $\boxed{u(x) \text{ is differentiable on } (0, 2\pi)}$

Since the hypotheses of the MVT are satisified by $u(x)$ on $[0, 2\pi]$, the theorem indicates that there exists at least one number c on the interval $(0, 2\pi)$ such that

$$u'(c) = \frac{u(2\pi) - u(0)}{2\pi - 0}$$

$$u'(x) = 1 - 3\cos^2 x \cdot (-\sin x) = 1 + 3\cos^2 x \sin x$$
$$u'(c) = 1 + 3\cos^2 c \, \sin c$$

$$u(0) = x - \cos^3(0) = -1$$
$$u(2\pi) = x - \cos^3(2\pi) = 2\pi - 1$$

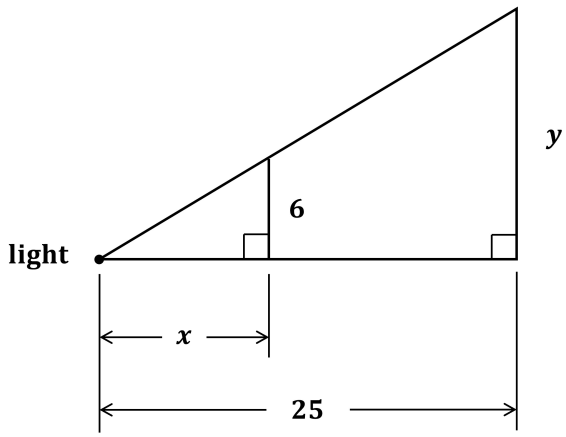
$$1 + 3\cos^2 c \, \sin c = \frac{(2\pi - 1) - (-1)}{2\pi} = 1$$
$$3\cos^2 c \, \sin c = 0$$

$$\cos c = 0 \quad \Rightarrow \quad c = \pi/2, 3\pi/2$$
$$\sin c = 0 \quad \Rightarrow \quad c = \pi$$

Therefore, the values of c on $(0, 2\pi)$ that satisfy the conclusion of the MVT are $\boxed{c = \pi/2, \pi, 3\pi/2}$

7. (20 pts) A spotlight on the ground shines on a wall 25 feet away. If a man who is 6 feet tall walks from the wall toward the spotlight at a constant speed of 4 feet per second, how fast is the height of his shadow increasing when he is 10 feet from the light? Include the correct unit of measurement in your answer.

Solution:



Let x represent the distance between the man and the light and let y represent the corresponding height of the man's shadow on the wall behind him, as depicted above. The two right triangles in the figure are similar so that the following relationship holds:

$$\frac{6}{x} = \frac{y}{25} \quad \Rightarrow \quad xy = 150$$

Next, apply implicit differentiation:

$$\frac{d}{dt}[xy] = \frac{d}{dt}[150]$$

$$x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt} = 0$$

Since the man travels toward the light at a constant speed of 4 feet per second, $\dfrac{dx}{dt} = -4$ ft/s.

When the man's distance from the light is $x = 10$ ft, the height of his shadow is $y = \dfrac{150}{10} = 15$ ft.

Therefore,

$$10 \cdot \frac{dy}{dt} + 15 \cdot (-4) = 0$$

$$\frac{dy}{dt} = \frac{60}{10} = \boxed{6 \text{ ft/s}}$$

8. (20 pts) Consider the function $w(x)$, defined as follows:

$$w(x) = \begin{cases} 2\sqrt{x} & , \quad x < 1 \\ x^2 - x + 2 & , \quad x \geq 1 \end{cases}$$

Is $w(x)$ differentiable at $x = 1$? Fully justify your answer using the limit definition of a derivative.

Note: $w(x)$ is continuous at $x = 1$; you do not have to prove its continuity.

Solution:

$w(x)$ differentiable at $x = 1$ if, and only if, $\displaystyle\lim_{x \to 1^-} \frac{w(x) - w(1)}{x - 1} = \lim_{x \to 1^+} \frac{w(x) - w(1)}{x - 1}$.

$$w(1) = 1^2 - 1 + 2 = 2$$

$$\begin{aligned}
\lim_{x \to 1^-} \frac{w(x) - w(1)}{x - 1} &= \lim_{x \to 1^-} \frac{2\sqrt{x} - 2}{x - 1} \\
&= \lim_{x \to 1^-} \frac{2(\sqrt{x} - 1)}{x - 1} \\
&= \lim_{x \to 1^-} \frac{2(\sqrt{x} - 1)}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \\
&= \lim_{x \to 1^-} \frac{2(x - 1)}{(x - 1)(\sqrt{x} + 1)} \\
&= \lim_{x \to 1^-} \frac{2}{\sqrt{x} + 1} = 1
\end{aligned}$$

$$\begin{aligned}
\lim_{x \to 1^+} \frac{w(x) - w(1)}{x - 1} &= \lim_{x \to 1^+} \frac{(x^2 - x + 2) - 2}{x - 1} \\
&= \lim_{x \to 1^+} \frac{x(x - 1)}{x - 1} \\
&= \lim_{x \to 1^+} x = 1
\end{aligned}$$

Therefore, since $\displaystyle\lim_{x \to 1^-} \frac{w(x) - w(1)}{x - 1} = \lim_{x \to 1^+} \frac{w(x) - w(1)}{x - 1}$, $\boxed{w(x) \text{ is differentiable at } x = 1}$