1. The following two problems are not related.

   (a) (12 pts) Evaluate $\int \sin x \cos x \, e^{\sin x}\, dx$. (*Hint:* Begin by making a substitution.)

   Solution: Let $u = \sin x$, $du = \cos x\, dx$.

$$\begin{aligned}\int \sin x \cos x \, e^{\sin x}\, dx &= \int \underbrace{u}_{\substack{U=u\\ dU=du}} \underbrace{e^u\, du}_{\substack{dV=e^u\,du\\ V=e^u}} \\ &\overset{IBP}{=} ue^u - \int e^u\, du \\ &= ue^u - e^u + C \\ &= \boxed{\sin x\, e^{\sin x} - e^{\sin x} + C}\end{aligned}$$

   (b) (10 pts) Let $I = \int_{100}^{k} \frac{1}{(x-k)^3}\, dx$ where k is a constant. Evaluate I. Specify the values of k for which I is convergent.

   Solution:

   There will be a vertical asymptote at $x = k$. Suppose $k > 100$. Then

$$\begin{aligned} I &= \int_{100}^{k} \frac{1}{(x-k)^3}\, dx \\ &= \lim_{t\to k^-} \int_{100}^{t} \frac{1}{(x-k)^3}\, dx \\ &= \lim_{t\to k^-} \left[-\frac{1}{2(x-k)^2} \right]_{100}^{t} \\ &= \lim_{t\to k^-} \left(-\frac{1}{2(t-k)^2} + \frac{1}{2(100-k)^2} \right) \\ &= \boxed{-\infty} \end{aligned}$$

   because $t - k$ will approach 0. If $k < 100$, the limit will approach k from the right but the result will again be $-\infty$. At $k = 100$ the integral is not well-defined. Therefore I $\boxed{\text{does not converge for any } k}$.

2. (22 pts) Determine whether the following expressions are convergent or divergent. If convergent, find the value the expression converges to. Be sure to fully justify your answers.

   (a) $a_n = n \sin\left(\frac{\pi}{n}\right)$ (b) $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^{n/2}}$ (c) $\sum_{n=2}^{\infty} \frac{6^{n-1}}{5^n (\ln n)^{10}}$

   Solution:

(a)

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \underbrace{n\sin\left(\tfrac{\pi}{n}\right)}_{\infty\cdot 0} \\
&= \lim_{n\to\infty} \underbrace{\frac{\sin\left(\frac{\pi}{n}\right)}{1/n}}_{0/0} \\
&\stackrel{LH}{=} \lim_{n\to\infty} \frac{\cos\left(\frac{\pi}{n}\right)\left(-\frac{\pi}{n^2}\right)}{-1/n^2} \\
&= \lim_{n\to\infty} \pi\cos\left(\tfrac{\pi}{n}\right) \\
&= \pi\cdot\cos 0 = \pi
\end{aligned}$$

The sequence a_n $\boxed{\text{converges to } \pi}$.

(b)

$$\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^{n/2}} = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{\ln 2}}\right)^n$$

This is a geometric series with ratio $\dfrac{1}{\sqrt{\ln 2}}$. Note that

$$2 < e \implies \ln 2 < \ln e = 1 \implies \sqrt{\ln 2} < 1 \implies \frac{1}{\sqrt{\ln 2}} > 1.$$

Because the ratio is greater than 1, the series $\boxed{\text{diverges}}$.

(c) Apply the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{6^n}{5^{n+1}(\ln(n+1))^{10}}\cdot\frac{5^n(\ln n)^{10}}{6^{n-1}}\right| \\
&= \lim_{n\to\infty} \frac{6^n}{6^{n-1}}\cdot\frac{5^n}{5^{n+1}}\cdot\left(\frac{\ln n}{\ln(n+1)}\right)^{10} \\
&\stackrel{LH}{=} \frac{6}{5}\left(\lim_{n\to\infty}\frac{1/n}{1/(n+1)}\right)^{10} \\
&= \frac{6}{5}\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^{10} \\
&\stackrel{LH}{=} \frac{6}{5}\cdot 1 = \frac{6}{5}
\end{aligned}$$

Because the ratio $\frac{6}{5} > 1$, the series $\boxed{\text{diverges}}$.

3. The following two problems are not related.

   (a) (8 pts) The geometric series $2 + \dfrac{8}{m} + \dfrac{32}{m^2} + \cdots$ has a sum of 5. What is the value of m?

   (b) (8 pts) A power series $\displaystyle\sum_{n=0}^{\infty} c_n(x+3)^n$ converges for $x = 1$ and diverges for $x = 20$. Find the minimum and maximum possible values for the radius of convergence.

Solution:

(a) The geometric series has a first term of $a = 2$ and a common ratio of $\frac{4}{m}$. The sum of the series is

$$S = \frac{a}{1-r} = \frac{2}{1-\frac{4}{m}} = 5.$$

Solving for m gives

$$5 - \frac{20}{m} = 2 \implies 3 = \frac{20}{m} \implies m = \boxed{\frac{20}{3}}.$$

(b) The power series is centered at $a = -3$ and converges for $x = 1$, so the radius of convergence $R \geq 4$. The series diverges for $x = 20$, so $R \leq 23$. The minimum and maximum values for R are $\boxed{4}$ and $\boxed{23}$, respectively.

4. (26 pts) The Maclaurin series for a function $f(x)$ is $\displaystyle\sum_{n=0}^{\infty} \frac{5^{n-1}}{n!} x^{n+1}$.

(a) Approximate the value of $f\left(\frac{1}{5}\right)$ using $T_2(x)$, the 2nd order Taylor polynomial.

(b) Use Taylor's Remainder Formula to find an error bound for the approximation found in part (a). Express your answer in terms of e. *Hint:* $f''(x) = e^{5x}(5x+2)$.

(c) Find the sum of the Maclaurin series.

Solution:

(a) The 2nd order Taylor polynomial is

$$T_2(x) = \frac{5^{-1}}{0!}x + \frac{5^0}{1!}x^2 = \frac{x}{5} + x^2.$$

An approximation for $f\left(\frac{1}{5}\right)$ is

$$T_2\left(\frac{1}{5}\right) = \frac{1/5}{5} + \left(\frac{1}{5}\right)^2 = \boxed{\frac{2}{25}}.$$

(b)

$$f''(x) = e^{5x}(5x+2)$$
$$f^{(3)}(x) = 5e^{5x} + 5e^{5x}(5x+2) = 5e^{5x}(5x+3)$$

$$R_2(x) = \frac{f^{(3)}(z)}{3!}x^3 \quad \text{for } 0 < z < \tfrac{1}{5}$$

Because $\left|f^{(3)}(z)\right|$ is an increasing function, it will be maximized at the right endpoint of the interval. The value of $f^{(3)}\left(\frac{1}{5}\right)$ is $5e(4) = 20e$. Therefore an error bound for the approximation is

$$|R_2(x)| < \frac{20e}{3!} \cdot \left(\frac{1}{5}\right)^3 = \frac{20e}{750} = \boxed{\frac{2e}{75}}.$$

(c) The series sums to

$$\sum_{n=0}^{\infty} \frac{5^{n-1}}{n!} x^{n+1} = \sum_{n=0}^{\infty} \frac{x}{5} \cdot \frac{5^n}{n!} x^n = \frac{x}{5} \sum_{n=0}^{\infty} \frac{(5x)^n}{n!} = \boxed{\frac{x}{5} e^{5x}} \quad \text{for all } x.$$

5. (28 pts) Let $\mathcal{R}$ be the region in the 1st quadrant bounded by the curve $y = 2\sqrt{1-x^2}$ and the positive x and y axes.

   (a) Sketch region $\mathcal{R}$. Label intercepts.

   (b) Set up (but do not evaluate) integrals to find the following quantities. Calculate any derivatives but otherwise integrands may be left unsimplified.

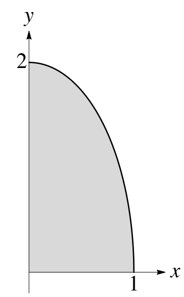
      i. Volume of the solid generated by rotating $\mathcal{R}$ about the x-axis using the Disk/Washer method.

      ii. Volume of the solid generated by rotating $\mathcal{R}$ about the x-axis using the Shell method.

      iii. Length of the curve in the 1st quadrant.

Solution:

(a) The curve is a quarter of the ellipse $y^2 = 4\left(1-x^2\right) \implies x^2 + \dfrac{y^2}{4} = 1$.



(b) i. Using the disk method, the volume is

$$V = \int_a^b \pi r^2\,dx = \boxed{\int_0^1 \pi\left(2\sqrt{1-x^2}\right)^2\,dx}.$$

ii. The curve can be represented as $x = \sqrt{1-\frac{y^2}{4}}$. Using the shell method, the volume is

$$V = \int_a^b 2\pi r h\,dy = \boxed{\int_0^2 2\pi y\sqrt{1-\frac{y^2}{4}}\,dy}.$$

iii. $y = 2\sqrt{1-x^2} \implies y' = \dfrac{2(-2x)}{2\sqrt{1-x^2}} = \dfrac{-2x}{\sqrt{1-x^2}}$. The arc length is

$$L = \int_a^b \sqrt{1+(y')^2}\,dx = \boxed{\int_0^1 \sqrt{1+\frac{4x^2}{1-x^2}}\,dx}.$$

6. The following two problems are not related.

   (a) (6 pts) A parametric representation for the curve $x^2 + \dfrac{y^2}{4} = 1$ in the 1st quadrant can be found by letting $x = \cos(\ln t)$. Find the corresponding equation for y in terms of t for $\alpha \le t \le \beta$.

   (b) (14 pts) Consider the curve $x = \ln(t+3)$, $y = (t+3)^2$.

      i. Use the parametric formula to find $\dfrac{d^2y}{dx^2}$ at $x = \ln 2$.

      ii. Find a Cartesian representation for the curve. Fully simplify and write your answer in the form $y = f(x)$.

Solution:

(a) To satisfy the identity $\cos^2\theta + \sin^2\theta = 1$ with $\theta = \ln t$, let $\boxed{y = 2\sin(\ln t)}$. Then

$$(\cos(\ln t))^2 + \frac{(2\sin(\ln t))^2}{4} = \cos^2(\ln t) + \sin^2(\ln t) = 1.$$

In the 1st quadrant, the angle $\theta = \ln t$ ranges from 0 to $\pi/2$. Solving for the t interval gives

$$\begin{gathered} 0 \le \ln t \le \frac{\pi}{2} \\ e^0 \le e^{\ln t} \le e^{\pi/2} \\ \boxed{1 \le t \le e^{\pi/2}}. \end{gathered}$$

(b) i.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{2(t+3)}{1/(t+3)} = 2(t+3)^2 \\ \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{4(t+3)}{1/(t+3)} = 4(t+3)^2 \end{aligned}$$

At $x = \ln 2$, the value of t is -1 and the slope is

$$\left.\frac{d^2y}{dx^2}\right|_{t=-1} = 4(2)^2 = \boxed{16}.$$

ii. $x = \ln(t+3) \implies t + 3 = e^x$. Substituting into the second equation gives

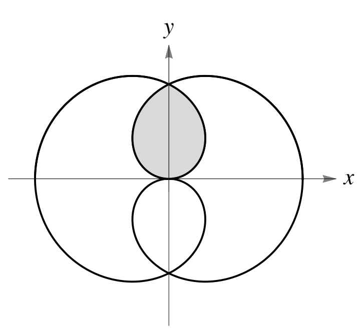
$$y = (e^x)^2 \implies \boxed{y = e^{2x}}.$$

7. (16 pts) Consider the polar curve $r = \cos(\theta/2)$.

(a) Set up (but do not evaluate) an integral to find the length of the curve.

(b) The curve contains two inner loops. Evaluate an integral to find the area inside one inner loop.

Solution:



(a) The curve is traced once on the interval $[0, 4\pi]$, so the arc length is

$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \boxed{\int_0^{4\pi} \sqrt{\cos^2\tfrac{\theta}{2} + \left(-\tfrac{1}{2}\sin\tfrac{\theta}{2}\right)^2}\, d\theta}.$$

(b) The inner loop above the x-axis is traced once on the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, so the area is

$$\begin{aligned}
\int_\alpha^\beta \tfrac{1}{2} r^2 \, d\theta &= \int_{\pi/2}^{3\pi/2} \tfrac{1}{2} \cos^2 \tfrac{\theta}{2} \, d\theta \\
&= \int_{\pi/2}^{3\pi/2} \tfrac{1}{2} \cdot \tfrac{1}{2} \left(1 + \cos\theta\right) \, d\theta \\
&= \left[\tfrac{1}{4}(\theta + \sin\theta)\right]_{\pi/2}^{3\pi/2} \\
&= \tfrac{1}{4}\left(\tfrac{3\pi}{2} - 1 - \left(\tfrac{\pi}{2} + 1\right)\right) \\
&= \tfrac{1}{4}\left(\pi - 2\right) = \boxed{\tfrac{\pi}{4} - \tfrac{1}{2}}.
\end{aligned}$$