1. [2360/121824 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

(a) An harmonic oscillator having a circular frequency of 2 sec^{-1} and mass equal to 2 kg must have a damping constant greater than 8 Nt/m/sec in order to be underdamped.

(b) $e^{12}\mathscr{L}\left\{e^{-3t}\text{step}(t-4)\right\} = \dfrac{e^{-4s}}{s+3}$

(c) The functions $1-x, 1+x$ and $1-3x$ form a basis for the solution space of $y''=0$ on $\mathbb{R}$.

(d) The set, $\mathbb{U}_{33}$, of 3×3 upper triangular matrices, is a subspace of $\mathbb{M}_{33}$ with dimension 6.

(e) For square matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ where $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}, \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ and $\mathbf{C}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ have only the trivial solution $\vec{\mathbf{x}} = \vec{\mathbf{0}}$, if $\mathbf{AB} = \mathbf{CA}$, then $|\mathbf{B}| = |\mathbf{C}|$.

(f) $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+2s+5}\right\} = e^{\pi - t}\sin 2t\,\text{step}(t-\pi)$

(g) The differential equation $\dot{x} = (x-1)\left(x^2+x-2\right)$ has a semistable equilibrium solution at $x=-2$.

(h) $-t^2\,\text{step}(t) + \left(t^2+2t-8\right)\text{step}(t-2) + (-2t+12)\,\text{step}(t-6) = \begin{cases} 0 & t<0 \\ -t^2 & 0\le t<2 \\ 2t-8 & 2\le t<6 \\ 4 & t\ge 6 \end{cases}$

SOLUTION:

(a) **FALSE** $\omega_0 = 2 = \sqrt{\dfrac{k}{2}} \Rightarrow 4 = \dfrac{k}{2} \Rightarrow k=8$. Then $b^2-4mk = b^2-4(2)(8) < 0 \Rightarrow b^2 < 64 \Rightarrow |b| < 8 \Rightarrow b<8$ since $b>0$.

(b) **TRUE** $e^{12}\mathscr{L}\left\{e^{-3t}\text{step}(t-4)\right\} = \mathscr{L}\left\{e^{-3(t-4)}\text{step}(t-4)\right\} = e^{-4s}\mathscr{L}\left\{e^{-3t}\right\} = \dfrac{e^{-4s}}{s+3}$

(c) **FALSE** The functions are all solutions of a linear homogeneous DE, but $W(1-x, 1+x, 1-3x) \equiv 0$ so they are linearly dependent and thus cannot form a basis. Note also that there are three vectors in vector space of dimension 2.

(d) **TRUE** The sum of two 3×3 upper triangular matrices is another 3×3 upper triangular matrix A as is the scalar multiple of a 3×3 upper triangular matrix. A basis for $\mathbb{U}_{33}$ is

$$\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$$

Since there are six linearly independent vectors in the set, the dimension of $\mathbb{U}_{3\times 3}$ is 6.

(e) **TRUE** The matrices are all invertible and thus have nonzero determinants so that

$$|\mathbf{AB}| = |\mathbf{CA}|$$
$$|\mathbf{A}||\mathbf{B}| = |\mathbf{C}||\mathbf{A}|$$
$$|\mathbf{B}| = |\mathbf{C}|$$

(f) **FALSE**

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+2s+5}\right\} &= \frac{1}{2}\text{step}(t-\pi)\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}\Bigg|_{t\to t-\pi} \\ &= \frac{1}{2}\text{step}(t-\pi)\left(e^{-t}\mathscr{L}^{-1}\frac{2}{s^2+4}\right)\Bigg|_{t\to t-\pi} \\ &= \frac{1}{2}\text{step}(t-\pi)e^{-(t-\pi)}\sin 2(t-\pi) \\ &= \frac{1}{2}e^{\pi-t}\sin 2t\,\text{step}(t-\pi)\end{aligned}$$

(g) **FALSE** Equilibrium solutions occur where $\dot{x} = (x-1)\left(x^2+x-2\right) = (x-1)^2(x+2) = 0$. This occurs when $x = 1, -2$ and we have $\dot{x} < 0$ if $x < -2$ and $\dot{x} > 0$ if $x > -2$ implying that $x=-2$ is unstable. $x=1$ is semistable.

(h) **TRUE** To see this, rewrite as $-t^2\,\text{step}(t) + t^2\,\text{step}(t-2) + (2t-8)\,\text{step}(t-2) - (2t-8)\,\text{step}(t-6) + 4\,\text{step}(t-6)$.

■

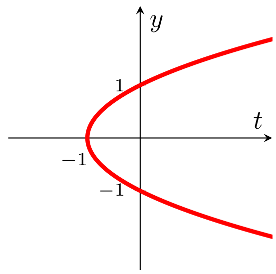
2. [2360/121824 (12 pts)] Consider the differential equation $y' - \dfrac{y^2}{t+1} = 0$.

   (a) (5 pts) Graph the set of points in the ty-plane where the slope of the solution is 1. Be sure to label any intercepts. With regard to the differential equation, what is the set of points called?

   (b) (7 pts) Find the explicit form of the solution of the differential equation passing through $\left(0, \frac{1}{4}\right)$.

SOLUTION:

(a) The set of points is called an isocline.



(b) The equation is separable.

$$\int y^{-2}\,\mathrm{d}y = \int \frac{1}{t+1}\,\mathrm{d}t$$

$$-\frac{1}{y} = \ln|t+1| + C \qquad \text{apply initial condition}$$

$$-4 = \ln|0+1| + C \implies C = -4$$

$$\frac{1}{y} = -\ln|t+1| + 4$$

$$y = \frac{1}{4 - \ln|t+1|} = \frac{1}{4 - \ln(t+1)} \qquad \text{since } t > -1$$

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3. [2360/121824 (15 pts)] Consider the linear algebraic system $\begin{array}{rl} x_1 + x_2 + x_3 &= 2 \\ 3x_1 + 6x_3 &= 15 \end{array}$.

   (a) (6 pts) Find a particular solution.

   (b) (4 pts) Find a basis for the solution space of the associated homogeneous system. What is the dimension of the solution space?

   (c) (3 pts) Use the Nonhomogeneous Principle to write the general solution to the system.

   (d) (2 pts) What is the rank of the coefficient matrix?

SOLUTION:

(a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 3 & 0 & 6 & 15 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 0 & 2 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & -1 & -3 \\ 1 & 0 & 2 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{array}\right] \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5-2t \\ -3+t \\ t \end{bmatrix},\ t \in \mathbb{R}$$

Choosing $t = 0$ gives a particular solution as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}$.

(b) The RREF for the associated homogeneous system is $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix}$, $t \in \mathbb{R}$. Thus a basis for the solution space is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ having dimension 1.

(c) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$

(d) rank $\mathbf{A} = 2$ since there are two pivot columns.

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4. [2360/121824 (18 pts)] Consider the initial value problem $ty' + 2y = \dfrac{\sin t}{t}$, $y(\pi) = 0$. Assume $t > 0$.

(a) (4 pts) Does Picard's Theorem guarantee that the IVP has a unique solution? Justify your answer.

(b) (4 pts) For any value of h, will a single step of Euler's method give the approximation $y(\pi + h) = 0$? Justify your answer.

(c) (10 pts) Solve the IVP.

SOLUTION:

(a) Yes. Rewrite the equation as $y' = \dfrac{\sin t}{t^2} - \dfrac{2y}{t}$ so that $f(t, y) = \dfrac{\sin t}{t^2} - \dfrac{2y}{t}$. Both $f(t, y)$ and $f_y(t, y) = -\dfrac{2}{t}$ are continuous for all t, y except $t = 0$. Thus a rectangle exists containing $(\pi, 0)$ on which both $f(t, y)$ and $f_y(t, y)$ are continuous, implying there exists $t_0 > 0$ such that the differential equation has a unique solution on $|t - \pi| < t_0$.

(b) Yes. One step of Euler's method is $y(\pi + h) \approx y_1 = y_0 + h\left(\dfrac{\sin t_0}{t_0^2} - \dfrac{2y_0}{t_0}\right) = 0 + h\left(\dfrac{\sin \pi}{\pi^2} - \dfrac{2(0)}{\pi}\right) = 0$

(c) Use the integrating factor method with the equation written as $y' + \dfrac{2}{t}y = \dfrac{\sin t}{t^2}$ so that $p(t) = 2/t$ and $\mu(t) = t^2$. Then

$$\int \left(t^2 y\right)' \,\mathrm{d}t = \int \sin t \,\mathrm{d}t$$

$$t^2 y = -\cos t + C \qquad \text{(apply the initial condition)}$$

$$\pi^2(0) = -\cos \pi + C \implies C = -1$$

$$y(t) = -\frac{1 + \cos t}{t^2}$$

Alternatively, use Euler-Lagrange Two Stage method (variation of parameters).

$$y_h' + \frac{2}{t} y_h = 0$$

$$\int \frac{\mathrm{d}y_n}{y_h} = -\int \frac{2}{t}\,\mathrm{d}t$$

$$\ln|y_h| = \ln|t|^{-2} + k \implies y_h = Ct^{-2}$$

$$y_p = v(t)t^{-2} \implies -2t^{-3}v(t) + t^{-2}v'(t) + 2t^{-3}v(t) = \frac{\sin t}{t^2}$$

$$\int v'(t)\,\mathrm{d}t = \int \sin t\,\mathrm{d}t \implies v(t) = -\cos t \implies y_p(t) = -\frac{\cos t}{t^2}$$

$$y = y_h + y_p = \frac{C}{t^2} - \frac{\cos t}{t^2}$$

$$y(\pi) = 0 = \frac{C}{\pi^2} + \frac{1}{\pi^2} \implies C = -1$$

$$y(t) = -\frac{1 + \cos t}{t^2}$$

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5. [2360/121824 (15 pts)] Solve the initial value problem $y' - y = t - 7\delta(t - 1) + te^t$, $y(0) = 3$.

SOLUTION:

$$sY(s) - y(0) - Y(s) = \frac{1}{s^2} - 7e^{-s} + \frac{1}{(s-1)^2}$$

$$Y(s) = \frac{1}{s^2(s-1)} - \frac{7e^{-s}}{s-1} + \frac{1}{(s-1)^3} + \frac{3}{s-1}$$

$$\text{partial fractions:} \quad \frac{1}{s^2(s-1)} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}$$

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{4}{s-1} - \frac{1}{s} - \frac{1}{s^2} - \frac{7e^{-s}}{s-1} + \frac{1}{(s-1)^3}\right\} \\ &= 4e^t - 1 - t - 7e^{t-1}\,\text{step}(t-1) + \frac{1}{2}t^2e^t \end{aligned}$$

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6. [2360/121824 (37 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} -1 & 1 & -3 \\ -2 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(a) (3 pts) Does $\mathbf{A}^{-1}$ exist? Justify your answer.

(b) (3 pts) Are the columns of $\mathbf{A}$ linearly dependent? Justify your answer.

(c) (5 pts) The characteristic equation of $\mathbf{A}$ is $\lambda^3 - 4\lambda^2 + 5\lambda = 0$. What are the eigenvalues of $\mathbf{A}$?

(d) (6 pts) Find the eigenvector associated with the real eigenvalue.

(e) (5 pts) Show that $\mathbf{A}\begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix} = (2+i)\begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix}$. Describe what this tells you.

(f) (15 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$, writing your answer as a single vector. Note: you have done much of the work for this in some of the previous parts of this problem.

SOLUTION:

(a) No.

$$|\mathbf{A}| = (-1)(-1)^{1+1}\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} + (1)(-1)^{1+2}\begin{vmatrix} -2 & -1 \\ 1 & 3 \end{vmatrix} + (-3)(-1)^{1+3}\begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} = (-1)(5) + (-1)(-5) + (-3)(0) = 0$$

Since $|\mathbf{A}| = 0$, the matrix is not invertible. Note that the RREF of $\mathbf{A}$ is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, which also serves as justification.

(b) Yes. Since $|\mathbf{A}| = 0$, the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has nontrivial solutions, meaning that there exist constants c_1, c_2, c_3, not all zero, such that

$$c_1\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For example, $c_1 = c_2 = 1$ and $c_3 = 0$. Note, as in part (a), the RREF of $\mathbf{A}$ can also be used as justification.

(c) $\lambda^3 - 4\lambda^2 + 5\lambda = \lambda\left(\lambda^2 - 4\lambda + 5\right) = 0$ so that $\lambda = 0$ and

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

(d) We need to find nontrivial solutions to $(\mathbf{A} - 0\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$

$$\left[\begin{array}{ccc|c} -1 & 1 & -3 & 0 \\ -2 & 2 & -1 & 0 \\ 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 1 & -3 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(e)

$$\mathbf{A}\begin{bmatrix}1\\i\\-1\end{bmatrix} = \begin{bmatrix}-1 & 1 & -3\\-2 & 2 & -1\\1 & -1 & 3\end{bmatrix}\begin{bmatrix}1\\i\\-1\end{bmatrix} = \begin{bmatrix}-1+i+3\\-2+2i+1\\1-i-3\end{bmatrix} = \begin{bmatrix}2+i\\-1+2i\\-2-i\end{bmatrix} \quad \text{and} \quad (2+i)\begin{bmatrix}1\\i\\-1\end{bmatrix} = \begin{bmatrix}2+i\\2i+i^2\\-2-i\end{bmatrix} = \begin{bmatrix}2+i\\-1+2i\\-2-i\end{bmatrix} \checkmark$$

This means that $2+i$ is an eigenvalue of $\mathbf{A}$ with eigenvector $\begin{bmatrix}1\\i\\-1\end{bmatrix}$.

(f) From the eigenvalue of 0, a solution of the system is $\vec{\mathbf{x}}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$. From parts (c) and (e), we know that the other eigenvalues are $2 \pm i$ with eigenvectors $\begin{bmatrix}1\\\pm i\\-1\end{bmatrix}$, respectively. We then have

$$\begin{bmatrix}1\\i\\-1\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix} + i\begin{bmatrix}0\\1\\0\end{bmatrix} = \vec{\mathbf{p}} + i\vec{\mathbf{q}}$$

and we write the other two solutions as

$$\vec{\mathbf{x}}_{\text{real}} = e^{2t}\left(\cos t\begin{bmatrix}1\\0\\-1\end{bmatrix} - \sin t\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = e^{2t}\begin{bmatrix}\cos t\\-\sin t\\-\cos t\end{bmatrix}$$

$$\vec{\mathbf{x}}_{\text{imag}} = e^{2t}\left(\sin t\begin{bmatrix}1\\0\\-1\end{bmatrix} + \cos t\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = e^{2t}\begin{bmatrix}\sin t\\\cos t\\-\sin t\end{bmatrix}$$

and the general solution as

$$\vec{\mathbf{x}}(t) = c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}\cos t\\-\sin t\\-\cos t\end{bmatrix} + c_3 e^{2t}\begin{bmatrix}\sin t\\\cos t\\-\sin t\end{bmatrix}$$

Applying the initial condition yields

$$\vec{\mathbf{x}}(0) = c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-2\\-2\end{bmatrix}$$

$$c_1 + c_2 = 0$$

$$c_1 + c_3 = -2$$

$$-c_2 = -2$$

which gives $c_2 = 2, c_1 = -2, c_3 = 0$ so that the solution to the IVP is $\begin{bmatrix}2e^{2t}\cos t - 2\\-2e^{2t}\sin t - 2\\-2e^{2t}\cos t\end{bmatrix}$

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7. [2360/121824 (14 pts)] Three 2000-ft^3 tanks are all initially half full with Tanks 1 and 3 containing fresh water and Tank 2 containing a solution in which 10 lb of contaminants are dissolved. The solutions in all tanks are always well-mixed. For $t > 0$, the flow rate in to Tank 1 and out of Tank 3 is 1 ft^3/hour while the flow rates from Tank 1 into Tank 2 and from Tank 2 into Tank 3 are both 2 ft^3/hour. Fresh water enters Tank 1 for $t < 5$ hours, after which solution containing $(2 + \cos t)$ lb/ft^3 of contaminants enters Tank 1. At precisely 10 hours, someone dumps 100 lb of contaminants into Tank 3.

   (a) (10 pts) Write, but **do not solve**, an initial value problem modeling the amounts (lb), $x_1(t), x_2(t), x_3(t)$, respectively, of contaminants in each tank. Write your answer using matrices and vectors.
   (b) (2 pts) Over what time interval is the solution valid?
   (c) (2 pts) Is the system autonomous or nonautonomous?

SOLUTION:

(a) Since the flow into and out of Tank 1 differ, the volume of solution in Tank 1 is governed by

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = \text{flow rate in} - \text{flow rate out} = 1 - 2 = -1,\ V_1(0) = 1000 \implies V_1(t) = 1000 - t$$

Similarly, for Tank 3,

$$\frac{\mathrm{d}V_3}{\mathrm{d}t} = \text{flow rate in} - \text{flow rate out} = 2 - 1 = 1,\ V_3(0) = 1000 \implies V_3(t) = 1000 + t$$

The volume in Tank 2 is steady at 1000 ft^3. We now apply the rule that the rate of change of the amount (mass) of contaminant in each tank equals the mass rate in minus the mass rate out.

Tank 1:

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \left(0\frac{\text{lb}}{\text{ft}^3}\right)\left(1\frac{\text{ft}^3}{\text{hour}}\right) + \left[(2+\cos t)\,\text{step}(t-5)\frac{\text{lb}}{\text{ft}^3}\right]\left(1\frac{\text{ft}^3}{\text{hour}}\right) - \left(\frac{x_1}{1000-t}\frac{\text{lb}}{\text{ft}^3}\right)\left(2\frac{\text{ft}^3}{\text{hour}}\right)\\ &= -\frac{2x_1}{1000-t} + (2+\cos t)\,\text{step}(t-5)\end{aligned}$$

Tank 2:

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \left(\frac{x_1}{1000-t}\frac{\text{lb}}{\text{ft}^3}\right)\left(2\frac{\text{ft}^3}{\text{hour}}\right) - \left(\frac{x_2}{1000}\frac{\text{lb}}{\text{ft}^3}\right)\left(2\frac{\text{ft}^3}{\text{hour}}\right) = \frac{2x_1}{1000-t} - \frac{x_2}{500}$$

Tank 3:

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = \left(\frac{x_2}{1000}\frac{\text{lb}}{\text{ft}^3}\right)\left(2\frac{\text{ft}^3}{\text{hour}}\right) + 100\delta(t-10) - \left(\frac{x_3}{1000+t}\frac{\text{lb}}{\text{ft}^3}\right)\left(1\frac{\text{ft}^3}{\text{hour}}\right) = \frac{x_2}{500} - \frac{x_3}{1000+t} + 100\delta(t-10)$$

We then have

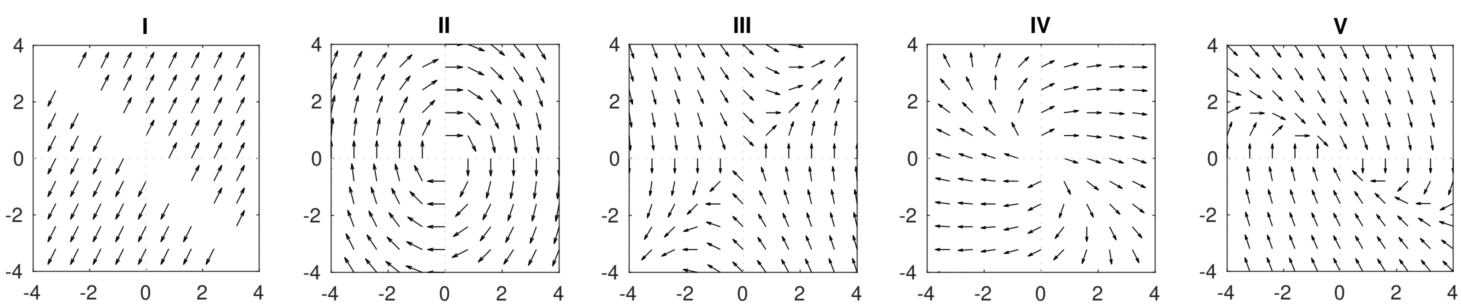
$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{1000-t} & 0 & 0 \\ \dfrac{2}{1000-t} & -\dfrac{1}{500} & 0 \\ 0 & \dfrac{1}{500} & -\dfrac{1}{1000+t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} (2+\cos t)\,\text{step}(t-5) \\ 0 \\ 100\delta(t-10) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

(b) The solution is valid until Tank 1 empties, that is, $[0, 1000)$. Note that this is the same time it takes for Tank 3 to completely fill.

(c) The system is nonautonomous.

8. [2360/121824 (15 pts)] Consider the following differential equations or systems of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ with the given matrix $\mathbf{A}$. Match the vector field to the appropriate system or equation and describe the geometry and stability of the fixed point(s). Write your answers in a well-organized table separate from any work you may do. No work need be shown.

(a) $y'' + 2y' + y = 0$ (b) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ (d) $y'' + 2y = 0$ (e) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$



SOLUTION:

For part (a) and (d), convert to a system of first order equations using $x_1 = y$ and $x_2 = y'$ yielding $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ for (a) and $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ for (d).

Part	Tr $\mathbf{A}$	$\lvert\mathbf{A}\rvert$	$(\text{Tr}\,\mathbf{A})^2 - 4\lvert\mathbf{A}\rvert$	geometry	stability	figure
(a)	-2	1	0	(attracting) degenerate node	asymptotically stable	V
(b)	3	0	9	nonisolated fixed points	unstable	I
(c)	4	5	-4	(repelling) spiral	unstable	IV
(d)	0	2	-8	center	neutrally stable	II
(e)	-1	-2	9	saddle	unstable	III