- This exam is worth 150 points and has 8 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"$\times$ 11" crib sheet with writing on both sides.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting. Remote students cannot interact with anyone except the proctor during the exam.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/121824 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) An harmonic oscillator having a circular frequency of 2 sec$^{-1}$ and mass equal to 2 kg must have a damping constant greater than 8 Nt/m/sec in order to be underdamped.

   (b) $e^{12}\mathscr{L}\left\{e^{-3t}\text{step}(t-4)\right\} = \dfrac{e^{-4s}}{s+3}$

   (c) The functions $1-x, 1+x$ and $1-3x$ form a basis for the solution space of $y''=0$ on $\mathbb{R}$.

   (d) The set, $\mathbb{U}_{33}$, of $3\times 3$ upper triangular matrices, is a subspace of $\mathbb{M}_{33}$ with dimension 6.

   (e) For square matrices $\mathbf{A},\mathbf{B},\mathbf{C}$ where $\mathbf{A}\vec{\mathbf{x}}=\vec{\mathbf{0}}, \mathbf{B}\vec{\mathbf{x}}=\vec{\mathbf{0}}$ and $\mathbf{C}\vec{\mathbf{x}}=\vec{\mathbf{0}}$ have only the trivial solution $\vec{\mathbf{x}}=\vec{\mathbf{0}}$, if $\mathbf{AB}=\mathbf{CA}$, then $|\mathbf{B}|=|\mathbf{C}|$.

   (f) $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+2s+5}\right\} = e^{\pi-t}\sin 2t\,\text{step}(t-\pi)$

   (g) The differential equation $\dot{x}=(x-1)\left(x^2+x-2\right)$ has a semistable equilibrium solution at $x=-2$.

   (h) $-t^2\,\text{step}(t)+\left(t^2+2t-8\right)\text{step}(t-2)+(-2t+12)\,\text{step}(t-6) = \begin{cases} 0 & t<0 \\ -t^2 & 0\le t<2 \\ 2t-8 & 2\le t<6 \\ 4 & t\ge 6 \end{cases}$

2. [2360/121824 (12 pts)] Consider the differential equation $y' - \dfrac{y^2}{t+1} = 0$.

   (a) (5 pts) Graph the set of points in the $ty$-plane where the slope of the solution is 1. Be sure to label any intercepts. With regard to the differential equation, what is the set of points called?

   (b) (7 pts) Find the explicit form of the solution of the differential equation passing through $\left(0,\frac{1}{4}\right)$.

3. [2360/121824 (15 pts)] Consider the linear algebraic system $\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 2 \\ 3x_1 & & & + & 6x_3 & = & 15 \end{array}$.

   (a) (6 pts) Find a particular solution.

   (b) (4 pts) Find a basis for the solution space of the associated homogeneous system. What is the dimension of the solution space?

   (c) (3 pts) Use the Nonhomogeneous Principle to write the general solution to the system.

   (d) (2 pts) What is the rank of the coefficient matrix?

4. [2360/121824 (18 pts)] Consider the initial value problem $ty'+2y=\dfrac{\sin t}{t},\ y(\pi)=0$. Assume $t>0$.

   (a) (4 pts) Does Picard's Theorem guarantee that the IVP has a unique solution? Justify your answer.

   (b) (4 pts) For any value of $h$, will a single step of Euler's method give the approximation $y(\pi+h)=0$? Justify your answer.

   (c) (10 pts) Solve the IVP.

5. [2360/121824 (15 pts)] Solve the initial value problem $y'-y=t-7\delta(t-1)+te^t,\ y(0)=3$.

**MORE PROBLEMS AND LAPLACE TRANSFORM TABLE BELOW/ON REVERSE**

6. [2360/121824 (37 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} -1 & 1 & -3 \\ -2 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(a) (3 pts) Does $\mathbf{A}^{-1}$ exist? Justify your answer.

(b) (3 pts) Are the columns of $\mathbf{A}$ linearly dependent? Justify your answer.

(c) (5 pts) The characteristic equation of $\mathbf{A}$ is $\lambda^3 - 4\lambda^2 + 5\lambda = 0$. What are the eigenvalues of $\mathbf{A}$?

(d) (6 pts) Find the eigenvector associated with the real eigenvalue.

(e) (5 pts) Show that $\mathbf{A} \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix} = (2+i) \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix}$. Describe what this tells you.

(f) (15 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$, writing your answer as a single vector. Note: you have done much of the work for this in some of the previous parts of this problem.

7. [2360/121824 (14 pts)] Three 2000-ft$^3$ tanks are all initially half full with Tanks 1 and 3 containing fresh water and Tank 2 containing a solution in which 10 lb of contaminants are dissolved. The solutions in all tanks are always well-mixed. For $t > 0$, the flow rate in to Tank 1 and out of Tank 3 is 1 ft$^3$/hour while the flow rates from Tank 1 into Tank 2 and from Tank 2 into Tank 3 are both 2 ft$^3$/hour. Fresh water enters Tank 1 for $t < 5$ hours, after which solution containing $(2 + \cos t)$ lb/ft$^3$ of contaminants enters Tank 1. At precisely 10 hours, someone dumps 100 lb of contaminants into Tank 3.

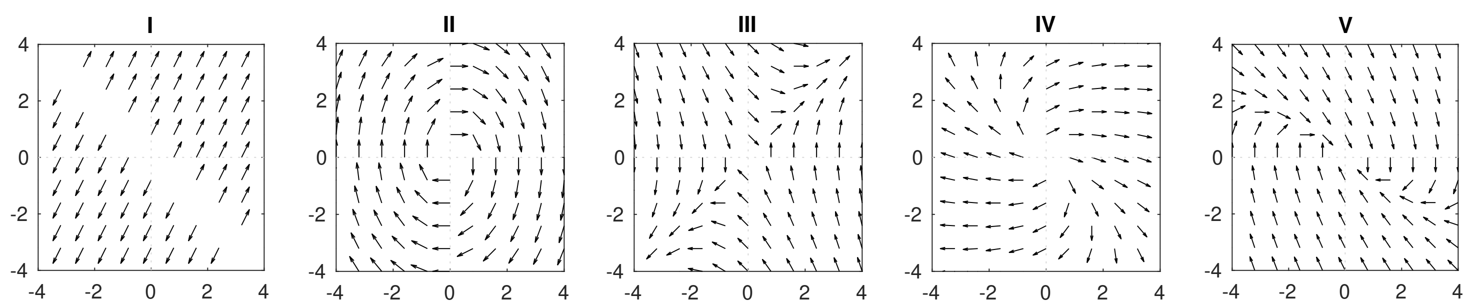
(a) (10 pts) Write, but **do not solve**, an initial value problem modeling the amounts (lb), $x_1(t), x_2(t), x_3(t)$, respectively, of contaminants in each tank. Write your answer using matrices and vectors.

(b) (2 pts) Over what time interval is the solution valid?

(c) (2 pts) Is the system autonomous or nonautonomous?

8. [2360/121824 (15 pts)] Consider the following differential equations or systems of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ with the given matrix $\mathbf{A}$. Match the vector field to the appropriate system or equation and describe the geometry and stability of the fixed point(s). Write your answers in a well-organized table separate from any work you may do. No work need be shown.

(a) $y'' + 2y' + y = 0$ (b) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ (d) $y'' + 2y = 0$ (e) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$




**Short table of Laplace Transforms:** $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,dt$

In this table, $a, b, c$ are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{dF(s)}{ds} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$