1. [2350/121824 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) The function $f(x,y) = \sqrt{x^2+y^2}$ has no critical points.

   (b) For any smooth path, $\mathbf{r}(t)$, in $\mathbb{R}^3$, the volume of the parallelepiped formed by $\mathbf{T}, \mathbf{N}, \mathbf{B}$ is 1.

   (c) The first degree Taylor polynomial centered at the origin of any function of the form $f(x,y) = ax+by+c$ is $f(x,y)$.

   (d) The graph of the equation $\rho^2 - 2\rho\sin\phi\cos\theta = 0$ is a sphere of radius 1 centered at $(0,1,0)$.

   (e) The line with symmetric equations $x - 4 = \dfrac{y+1}{-2} = z - 3$ never intersects the plane $x+y+z=1$.

   (f) Suppose that for a given function $f_x(x,y) = (x-1)^2 - y$ and $f_y(x,y) = y - x + 1$. Then $f(x,y)$ has a local maximum at $(2,1)$.

   (g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+y^3}$ does not exist.

   (h) The vertical traces of $z - y - x^2 = 0$ in planes parallel to the yz-plane are lines.

   SOLUTION:

   (a) **FALSE** $f_x = \dfrac{x}{\sqrt{x^2+y^2}}$ and $f_y = \dfrac{y}{\sqrt{x^2+y^2}}$ which both fail to exist at $(0,0)$, implying that $(0,0)$ is a critical point.

   (b) **TRUE** The volume is given by, for example, $|\mathbf{T}\cdot(\mathbf{N}\times\mathbf{B})| = |\mathbf{T}\cdot\mathbf{T}| = 1$

   (c) **TRUE** $f_x(0,0) = a$, $f_y(0,0) = b$, $f(0,0) = c$ and $T_1(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = c + ax + by = f(x,y)$.

   (d) **FALSE** It is a sphere of radius 1 centered at $(1,0,0)$. Convert to cartesian coordinates.

   $$\rho^2 - 2\rho\sin\phi\cos\theta = 0 \implies x^2+y^2+z^2-2x = (x-1)^2+y^2+z^2 = 1$$

   (e) **TRUE** The direction vector of the line is $\mathbf{v} = \langle 1,-2,1\rangle$ and the normal vector to the plane is $\mathbf{n} = \langle 1,1,1\rangle$ with $\mathbf{v}\cdot\mathbf{n} = 0$. These vectors are orthogonal, implying that the line is parallel to the plane but could lie in the plane, in which case the two would intersect. However, since $(4,-1,3)$ is on the line and not in the plane, there is no intersection.

   (f) **FALSE**

   $$f_y = 0 \implies x = y+1 \implies f_x = y(y-1) = 0 \implies y = 0,1 \implies x = 1,2, \text{ respectively} \implies (2,1),(1,0) \text{ critical points}$$
   $$f_{xx}(x,y) = 2(x-1) \implies f_{xx}(2,1) = 2$$
   $$f_{yy}(x,y) = 1 \implies f_{yy}(2,1) = 1$$
   $$f_{xy}(x,y) = -1 \implies f_{xx}(2,1)f_{yy}(2,1) - [f_{xy}(2,1)]^2 = 1 > 0 \text{ and } f_{xx}(2,1) > 0 \implies f(2,1) \text{ is a local minimum}$$

   (g) **TRUE**

   $$\lim_{(0,y)\to(0,0)} \frac{3(0)^2}{0^2+y^3} = 0 \neq 3 = \lim_{(x,0)\to(0,0)} \frac{3x^2}{x^2+0^3}$$

   (h) **TRUE** Planes parallel to the yz-plane have equations $x = x_0$ where x_0 is a constant. This gives $z - y = x_0^2$ which describes a line.

2. [2350/121824 (15 pts)] The area of a triangle is $A(a,b,C) = \dfrac{1}{2}ab\sin C$ where a and b are two sides of the triangle and C is the angle between the sides of length a and b. In surveying a particular triangular plot of land, the sides a and b are measured to be 120 and 200 feet, respectively, and C is read to be $\pi/3$ radians. By how much is the computed area in error if the sizes of each side are in error by 0.1 foot and C is in error by $\pi/60$ radians?

   SOLUTION:

We use differentials.

$$\mathrm{d}A = \frac{\partial A}{\partial a}\,\mathrm{d}a + \frac{\partial A}{\partial b}\,\mathrm{d}b + \frac{\partial A}{\partial C}\,\mathrm{d}C = \frac{1}{2}b\sin C\,\mathrm{d}a + \frac{1}{2}a\sin C\,\mathrm{d}b + \frac{1}{2}ab\cos C\,\mathrm{d}C$$

$$\mathrm{d}A = \frac{1}{2}(200)\frac{\sqrt{3}}{2}\left(\frac{1}{10}\right) + \frac{1}{2}(120)\frac{\sqrt{3}}{2}\left(\frac{1}{10}\right) + \frac{1}{2}(120)(200)\left(\frac{1}{2}\right)\left(\frac{\pi}{60}\right)$$

$$= 5\sqrt{3} + 3\sqrt{3} + 100\pi = 8\sqrt{3} + 100\pi \ \text{ ft}^2$$

■

3. [2350/121824 (17 pts)] Ralphie the buffalo is part of a herd on the eastern plains of Colorado containing $\delta(x, y) = 60y$ buffalo per square kilometer. A ranch on the plains is given by the region, $\mathcal{D}$, bounded by the x- and y-axes and the line $x + y = 1$ where distances are measured in kilometers.

   (a) (5 pts) Fully set up, but **do not evaluate**, an integral that computes the number of buffalo on the ranch.

   (b) (12 pts) Ralphie decides to lead a stampede of the buffaloes to Folsom Field with the velocity vector field of the herd given by

$$\mathbf{V} = \left(2x - y^4\right)\mathbf{i} + x^2\,\mathbf{j} \ \text{ km/hour}$$

   The flux vector field, $\mathbf{F} = \delta\mathbf{V}$, gives the number of buffaloes per hour per kilometer and can be used to determine the flux (buffaloes per hour) of the herd passing across a curve in the plane. Use one of the Fundamental Theorems that we studied this semester to find the net number of buffaloes per hour exiting across the boundary of the ranch (region $\mathcal{D}$).

SOLUTION:

(a)

$$N = \iint_{\mathcal{D}} \delta(x, y)\,\mathrm{d}A = \int_0^1 \int_0^{1-y} 60y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^{1-x} 60y\,\mathrm{d}y\,\mathrm{d}x$$

(b) We need to find the flux of $\mathbf{F}$ across the boundary, ∂D, of the region $\mathcal{D}$. This quantity is given by $\oint_{\partial\mathcal{D}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s$. We use Green's Theorem to compute this, that is,

$$\text{Flux} = \oint_{\partial\mathcal{D}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \iint_{\mathcal{D}} \nabla\cdot\mathbf{F}\,\mathrm{d}A = \iint_{\mathcal{D}} \nabla\cdot 60\langle 2xy - y^5, x^2y\rangle\,\mathrm{d}A$$

$$= 60\int_0^1 \int_0^{1-x} \left(2y + x^2\right)\mathrm{d}y\,\mathrm{d}x = 60\int_0^1 \left(y^2 + x^2y\right)\Big|_0^{1-x}\,\mathrm{d}x$$

$$= 60\int_0^1 \left(1 - 2x + x^2 + x^2 - x^3\right)\mathrm{d}x = 60\left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Bigg|_0^1 = 25 \text{ buffaloes per hour}$$

■

4. [2350/121824 (18 pts)] Some friends of yours are running laps in a counterclockwise direction around the unit circle in the presence of the force field $\mathbf{F} = \left(Axy - y^3\right)\mathbf{i} + \left(4y + 3x^2 - 3xy^2\right)\mathbf{j}$ where A is a constant.

   (a) (6 pts) Having gone from $(1, 0)$ to $(0, 1)$, they are complaining about how much work they have done. You tell them to stop whining because when they get back to $(1, 0)$ they will have done no work at all. Indeed, they could walk in any closed path and not do any work. What is A?

   (b) (12 pts) To get their minds off all the work they are doing, you ask them to walk along the path, $\mathcal{C}$, parameterized by

$$\mathbf{r}(t) = \left(2e^{t^2 - t} + t\right)\mathbf{i} + t\cos(\pi t)\,\mathbf{j},\ 0 \le t \le 1$$

   instead. Find the work done by the force field on your friends during that adventure.

SOLUTION:

(a) The fact that every closed path results in zero work means that the vector field is conservative. Therefore is has zero curl. Thus

$$\frac{\partial}{\partial x}\left(4y+3x^2-3xy^2\right)-\frac{\partial}{\partial y}\left(Axy-y^3\right)=0$$
$$6x-3y^2-(Ax-3y^2)=0$$
$$6x=Ax$$
$$A=6$$

(b) We need to compute $\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$. However, doing this directly will be nightmarish, perhaps impossible. However, we can use the Fundamental Theorem for Line Integrals to simplify the work by finding a potential function, f, such that $\mathbf{F}=\nabla f$.

$$\frac{\partial f}{\partial x}=6xy-y^3 \implies f(x,y)=\int\left(6xy-y^3\right)\mathrm{d}x=3x^2y-xy^3+g(y)$$
$$\frac{\partial f}{\partial y}=3x^2-3xy^2+g'(y)=4y+3x^2-3xy^2 \implies g'(y)=4y$$
$$g(y)=\int 4y\,\mathrm{d}y=2y^2+c \implies f(x,y)=3x^2y-xy^3+2y^2+c$$
$$\text{Work}=\int_{(2,0)}^{(3,-1)}\nabla f\cdot\mathrm{d}\mathbf{r}=f(3,-1)-f(2,0)=3(3^2)(-1)-3(-1)^3+2(-1)^2-0=-22$$

Alternatively, one could choose to integrate along a different path, $\mathcal{C}_1$, perhaps a straight line segment between the points, and then evaluate $\int_{\mathcal{C}_1}\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ directly. That involves more work (pun not intended).

■

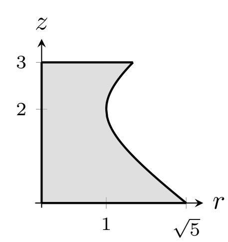
5. [2350/121824 (31 pts)] A cooling tower is in the shape of $(z-2)^2=x^2+y^2-1,\ 0\le z\le 3$. The vector field describing the movement of heat is $\mathbf{H}=\left\langle 2xz+y^2, e^{xz}, z^2\right\rangle$. Your mission is to compute the heat flux through the top, bottom and side of the tower, a closed surface.

(a) (3 pts) Name the surface.

(b) (4 pts) Without evaluating any integrals, describe in words why $\iint_{\mathcal{S}_b}\mathbf{H}\cdot\mathbf{n}\,\mathrm{d}S=0$, where $\mathcal{S}_b$ is the portion of the tower in the xy-plane.

(c) (12 pts) By direct calculation, find the upward heat flux through the top, $\mathcal{S}_t$, of the tower.

(d) (12 pts) With the aid of one of the Fundamental Theorems that we have studied, find the heat flux through the side, $\mathcal{S}$, of the tower. The following diagram shows a cross section (constant θ) through the tower.



SOLUTION:

(a) Rewriting, we have $x^2+y^2-(z-2)^2=1$, a hyperboloid of one sheet.

(b) There is no $\mathbf{k}$-component in the vector field in the xy-plane where $z=0$ so $\mathbf{H}\cdot\mathbf{n}=0$ since $\mathbf{n}=-\mathbf{k}$.

(c) The top of the tower is given by $g(x,y,z)=z=3 \implies \nabla g=\mathbf{k}$. Project onto the xy-plane so that $\mathbf{p}=\mathbf{k}$ and $|\nabla g\cdot\mathbf{p}|=1$ and the region of integration, $\mathcal{R}$, is $x^2+y^2\le 2$. For upward flux we use $+\nabla g$ for $\mathbf{n}$ and thus

$$\text{Flux}=\iint_{\mathcal{S}_t}\mathbf{H}\cdot\mathbf{n}\,\mathrm{d}S=\iint_{\mathcal{R}}\left\langle 2xz+y^2,e^{xz},z^2\right\rangle\cdot\langle 0,0,1\rangle\,\mathrm{d}A=\iint\limits_{x^2+y^2\le 2} z^2\,\mathrm{d}A=\iint\limits_{x^2+y^2\le 2} 9\,\mathrm{d}A=18\pi$$

(d) Let $\mathcal{E}$ be the solid region inside the tower with the boundary of $\mathcal{E}$ given by $\partial\mathcal{E} = \mathcal{S} \cup \mathcal{S}_t \cup \mathcal{S}_b$. Then Gauss' Divergence Theorem gives

$$\iiint_{\mathcal{E}} \nabla \cdot \mathbf{H} \, \mathrm{d}V = \iint_{\partial\mathcal{E}} \mathbf{H} \cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{S}} \mathbf{H} \cdot \mathrm{d}\mathbf{S} + \iint_{\mathcal{S}_t} \mathbf{H} \cdot \mathrm{d}\mathbf{S} + \iint_{\mathcal{S}_b} \mathbf{H} \cdot \mathrm{d}\mathbf{S}$$

Since this is a closed region, the outward pointing normal will be used. From parts (b) and (c) we know that $\iint_{\mathcal{S}_t} \mathbf{H} \cdot \mathrm{d}\mathbf{S} = 18\pi$ and $\iint_{\mathcal{S}_b} \mathbf{H} \cdot \mathrm{d}\mathbf{S} = 0$ so that $\iint_{\mathcal{S}} \mathbf{H} \cdot \mathrm{d}\mathbf{S} = -18\pi + \iiint_{\mathcal{E}} \nabla \cdot \mathbf{H} \, \mathrm{d}V$. We'll use cylindrical coordinates to evaluate the triple integral after noting that

$$\begin{aligned}
\nabla \cdot \left\langle 2xz + y^2, e^{xz}, z^2 \right\rangle &= 4z \\
\iiint_{\mathcal{E}} \nabla \cdot \mathbf{H} \, \mathrm{d}V &= \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{(z-2)^2+1}} 4zr \, \mathrm{d}r \, \mathrm{d}z \, \mathrm{d}\theta \\
&= 2 \int_0^{2\pi} \int_0^3 z \left(z^2 - 4z + 5 \right) \mathrm{d}z \, \mathrm{d}\theta \\
&= 2 \int_0^{2\pi} \int_0^3 \left(z^3 - 4z^2 + 5z \right) \mathrm{d}z \, \mathrm{d}\theta \\
&= 4\pi \left(\frac{z^4}{4} - \frac{4z^3}{3} + \frac{5z^2}{2} \right) \Bigg|_0^3 \\
&= 36\pi \left(\frac{9}{4} - 4 + \frac{5}{2} \right) = 36\pi \left(\frac{3}{4} \right) = 27\pi
\end{aligned}$$

Finally then $\iint_{\mathcal{S}} \mathbf{H} \cdot \mathrm{d}\mathbf{S} = -18\pi + 27\pi = 9\pi$.

Alternatively (ouch!):

$$\begin{aligned}
\iiint_{\mathcal{E}} \nabla \cdot \mathbf{H} \, \mathrm{d}V &= \int_0^{2\pi} \int_0^1 \int_0^3 4zr \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta + \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{2+\sqrt{r^2-1}}^3 4zr \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta + \int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{2-\sqrt{r^2-1}} 4zr \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta \\
&= 18\pi + \frac{11\pi}{3} + \frac{16\pi}{3} = 27\pi
\end{aligned}$$

■

6. [2350/121824 (27 pts)] Let $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j} - z^2\,\mathbf{k}$.

(a) (12 pts) Directly evaluate $\iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$, where $\mathcal{S}$ is the portion of the sphere of radius 2 centered at the origin below the plane $z = -1$ oriented with an upward pointing normal.

(b) (12 pts) Directly evaluate $\int_{\mathcal{C}} y \, \mathrm{d}x - x \, \mathrm{d}y - z^2 \, \mathrm{d}z$, where $\mathcal{C}$ is the circle of radius $\sqrt{3}$ centered at the origin in the plane $z = -1$ oriented counterclockwise when viewed from above.

(c) (3 pts) Why are the answers to parts (a) and (b) the same?

SOLUTION:

(a)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & z^2 \end{vmatrix} = -2\,\mathbf{k}$$

$$g(x,y,z) = x^2 + y^2 + z^2 \implies \nabla g = \langle 2x, 2y, 2z \rangle \quad (-\nabla g \text{ for proper orientation since } z < 0)$$

$$\text{project } \mathcal{S} \text{ onto } xy\text{-plane} \implies \mathbf{p} = \mathbf{k} \text{ and } \mathcal{R} : x^2 + y^2 \le 3 \text{ and } |\nabla g \cdot \mathbf{p}| = |2z| = -2z \text{ since } z < 0$$

$$\nabla \times \mathbf{F} \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|} = \langle 0, 0, -2 \rangle \cdot \frac{\langle -2x, -2y, -2z \rangle}{-2z} = -2$$

$$\iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} -2\,\mathrm{d}A = \iint\limits_{x^2+y^2 \le 3} -2\,\mathrm{d}A = -2\,\text{area}(\mathcal{R}) = -2\pi\left(\sqrt{3}\right)^2 = -6\pi$$

(b) $\mathcal{C}$ can be parameterized as $(x(t), y(t), z(t)) = \left(\sqrt{3}\cos t, \sqrt{3}\sin t, -1\right),\ 0 \le t \le 2\pi$. Then

$$\mathrm{d}x = -\sqrt{3}\sin t\,\mathrm{d}t, \quad \mathrm{d}y = \sqrt{3}\cos t\,\mathrm{d}t, \quad \mathrm{d}z = 0\,\mathrm{d}t$$

$$\int_{\mathcal{C}} y\,\mathrm{d}x - x\,\mathrm{d}y - z^2\,\mathrm{d}z = \int_0^{2\pi} \left[\left(\sqrt{3}\sin t\right)\left(-\sqrt{3}\sin t\right) - \left(\sqrt{3}\cos t\right)\left(\sqrt{3}\cos t\right)\right]\mathrm{d}t = \int_0^{2\pi} -3\,\mathrm{d}t = -6\pi$$

(c) The curve, $\mathcal{C}$, in part (b) represents the boundary of the surface, $\mathcal{S}$, in part (a). The integrals represent the two sides of Stokes's theorem and thus must be equal.

■

7. [2350/121824 (18 pts)] You have just purchased a piece of land on which to build a ski resort. The piece of land can be described as the region in the xy-plane satisfying the inequalities $|x| \le 10$ and $|y| \le 9$. The elevation of the terrain in the resort is

$$f(x,y) = x - \frac{1}{12}x^3 - \frac{1}{4}y^2 + 6$$

(a) (4 pts) You want to place some weather instruments on the highest point and a restaurant at the lowest point in the ski resort. **Without** doing any calculations, can this be accomplished? Why or why not?

(b) (7 pts) The midpoint of one of the ski runs will be above the point $(x,y) = (4,2)$. Find a unit vector giving the direction that will produce the "most downhill" (*i.e.* "most negative", minimum) slope of the run at that point. What will the minimum slope be?

(c) (7 pts) Suppose you are cross-country skiing on the path whose projection on the xy-plane is $\mathbf{r}(t) = t\,\mathbf{i} + \dfrac{t^2}{2}\,\mathbf{j}$, $-4 \le t \le 4$. Use an appropriate Calculus 3 chain rule to determine your rate of change of elevation when you are at the point $\left(1, \frac{1}{2}, \frac{329}{48}\right)$.

SOLUTION:

(a) Yes. The ski area's elevation is a continuous function throughout $\mathbb{R}^2$ and the ski resort is a closed, bounded set. Thus the Extreme Value Theorem guarantees the existence of a highest and lowest point in the ski resort.

(b) We need the negative of the gradient at the point.

$$-\nabla f(x,y) = -\left\langle 1 - \frac{1}{4}x^2, -\frac{1}{2}y \right\rangle \implies -\nabla f(4,2) = -\langle -3, -1 \rangle = \langle 3, 1 \rangle \qquad \text{unit vector is } \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$$

The minimum slope will be $-\|\nabla(4,2)\| = -\sqrt{10}$.

(c) The rate of change of elevation is $\dfrac{\mathrm{d}f}{\mathrm{d}t} = \nabla f(x,y) \cdot \mathbf{r}'(t)$. You will arrive at the point in question when $t = 1$ and $\mathbf{r}'(t) = \langle 1, t \rangle$.

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=1} = \nabla f\left(1, \tfrac{1}{2}\right) \cdot \mathbf{r}'(1) = \left\langle \frac{3}{4}, -\frac{1}{4} \right\rangle \cdot \langle 1, 1 \rangle = \frac{1}{2}$$

■