1. [2360/112024 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) If the equation of motion for a certain undamped mass/spring harmonic oscillator having a spring(restoring) constant of 2 N/m is $x(t) = 10\cos\left(\frac{t}{2}\right) - 5\sin 2t$, then the mass is 8 kg.

   (b) The differential equation $y''' - t = 0$ can be written as a system of three first order equations in the form $\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & t \end{bmatrix} \vec{\mathbf{x}}$.

   (c) $\ddot{x} - tx = 0$ describes a conservative system.

   (d) The solution space of $y^{(4)} + 8y'' + 16y = 0$ is span $\{\cos 2t, \sin 2t, t\cos 2t, t\sin 2t\}$.

   (e) Let $L(\vec{\mathbf{y}}) = f(t)$ where L represents a linear differential operator with constant coefficients. If the characteristic equation of $L(\vec{\mathbf{y}}) = 0$ is $r^3(r-1)^2 = 0$ and $f(t) = 3t\left(e^t - 2t^2\right)$, then the guess for the particular solution when using the method of undetermined coefficients is $y_p = \left(At^3 + Bt^2\right)e^t + Ct^3 + Dt^2 + Et + F$.

   SOLUTION:

   (a) **FALSE** $\omega_0 = \frac{1}{2} = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{m}} \implies m = 8$ or $\omega_0 = 2 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{m}} \implies m = \frac{1}{2}$.

   (b) **FALSE** The correct system is $\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$.

   (c) **FALSE** The equation is not autonomous.

   (d) **TRUE** $r^4 + 8r^2 + 16 = \left(r^2 + 4\right)^2 = 0 \implies r = \pm 2i$ each with multiplicity 2.

   (e) **FALSE** The basis for the solution space of the associated homogeneous equation is $\left\{e^t, te^t, 1, t, t^2\right\}$ so the correct guess is $y_p = \left(At^3 + Bt^2\right)e^t + Ct^6 + Dt^5 + Et^4 + Ft^3$

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2. [2360/112024 (12 pts)] Consider the ordinary differential equation $y''' = 9y'$.

   (a) (6 pts) Determine the general solution of the differential equation.

   (b) (6 pts) Verify that the solutions found in part (a) are linearly independent.

   SOLUTION:

   (a) We assume solutions of the form $y = e^{rt}$. Substituting this into the differential equation yields the characteristic polynomial $r^3 - 9r = r(r^2 - 9) = r(r+3)(r-3)$. General solution is $y(t) = c_1 + c_2 e^{3t} + c_3 e^{-3t}$.

   (b) We verify linear independence using the Wronskian of the solutions.

$$W\left[1, e^{3t}, e^{-3t}\right] = \begin{vmatrix} 1 & e^{3t} & e^{-3t} \\ 0 & 3e^{3t} & -3e^{-3t} \\ 0 & 9e^{3t} & 9e^{-3t} \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 3e^{3t} & -3e^{-3t} \\ 9e^{3t} & 9e^{-3t} \end{vmatrix} = \left(3e^{3t}\right)\left(9e^{-3t}\right) - \left(9e^{3t}\right)\left(-3e^{-3t}\right) = 54 \neq 0$$

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3. [2360/112024 (12 pts)] You received a toy spring with a 0.25 kg mass attached to it as a recent birthday present. Recognizing the toy as a harmonic oscillator and being mathematically inclined you, of course, have made some measurements:

   - 8 N are required to stretch the spring 2 m
   - when moving at 6 m/s, the mass experiences a damping force of 12 N

   (a) (6 pts) One day, while playing with the toy, you pull the mass $\frac{1}{2}$ m to the right of its equilibrium position and then push it to the left at 3 m/sec. Write, but **do not solve**, the initial value problem governing the displacement, $x(t)$, of the oscillator if an external force of te^{-t} is acting on it.

   (b) (6 pts) Without finding $x(t)$ in part (a), answer the following.

i. (3 pts) Providing mathematical justification, determine whether the oscillator is underdamped, overdamped or critically damped.

ii. (3 pts) Will you get the thrill of watching the mass pass through the equilibrium position twice? Why or why not?

SOLUTION:

(a)

$$F = -kx \implies -8 = -k(2) \implies k = 4 \text{ N/m}$$
$$F = -b\dot{x} \implies -12 = -b(6) \implies b = 2 \text{ N/m/s}$$
$$\frac{1}{4}\ddot{x} + 2\dot{x} + 4x = te^{-t}, \quad x(0) = \frac{1}{2}, \quad \dot{x}(0) = -3$$

i. $b^2 - 4mk = 2^2 - 4\left(\frac{1}{4}\right)(4) = 0 \implies$ critically damped

ii. No. Since the oscillator is critically damped, it will pass through the equilibrium position at most once.
Remark: Technically, a critically damped, <u>unforced</u> oscillator is guaranteed to pass through the origin only once (this was the spirit of the question). This may not be the case for a forced one, depending on the forcing function. How could you determine if, indeed, a critically damped, forced oscillator only passes through the equilibrium position once?

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4. [2360/112024 (20 pts)] Consider the differential equation $x^2z'' - xz' = x^2 \ln x, \; x > 0$.

(a) (8 pts) Assuming solutions of the form $z = x^r$, find a basis for the solution space of the associated homogeneous equation.

(b) (12 pts) Find the general solution of the original, nonhomogeneous equation.

SOLUTION:

(a)

$$z = x^r \implies z' = rx^{r-1} \implies z'' = r(r-1)x^{r-2}$$
$$x^2z'' - xz' = x^2\left(r^2 - r\right)x^{r-2} - xrx^{r-1} = (r^2 - 2r)x^r = 0 \implies r(r-2) = 0 \implies r = 0, 2$$

Basis for the solution space of the associated homogeneous equation is $\left\{1, x^2\right\}$.

(b) Standard form for DE is $z'' - \frac{z'}{x} = \ln x$. Let $z_1 = 1, z_2 = x^2$ and $z_p = v_1z_1 + v_2z_2$.

$$W[z_1, z_2] = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x$$
$$v_1 = -\int \frac{x^2 \ln x}{2x}\,\mathrm{d}x = -\frac{1}{2}\int x \ln x\,\mathrm{d}x \overset{\text{IBP}}{=} -\frac{x^2}{8}\left(2\ln x - 1\right)$$
$$v_2 = \int \frac{\ln x}{2x}\,\mathrm{d}x \overset{u=\ln x}{=} \frac{(\ln x)^2}{4}$$
$$z_p = -\frac{x^2}{8}\left(2\ln x - 1\right) + \left(\frac{x \ln x}{2}\right)^2$$
$$\text{general solution:} \qquad z(x) = c_1 + c_2x^2 - \frac{x^2}{8}\left(2\ln x - 1\right) + \left(\frac{x\ln x}{2}\right)^2$$

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5. [2360/112024 (26 pts)] Consider the harmonic oscillator described by the initial value problem

$$\ddot{x} + 2\dot{x} + 10x = 50t, \quad x(0) = 1, \quad \dot{x}(0) = -3$$

(a) (20 pts) Find the transient and steady state solutions, if any exist.

(b) (6 pts) Now suppose the damping is removed and the oscillator is given a driving force of the form $F_0 \cos \omega_f t$.

i. (3 pts) Find the frequency, ω_f, that will put the system into pure resonance.

ii. (3 pts) If you were to solve the differential equation modeling this new situation with the ω_f from part (i) (do not do so), what would be the **form** of the particular solution?

SOLUTION:

(a)

$$r^2 + 2r + 10 = 0$$

$$r = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = -1 + 3i$$

$$x_h = e^{-t}\left(c_1 \cos 3t + c_2 \sin 3t\right)$$

$$x_p = At + B$$

$$\ddot{x}_p + 2\dot{x}_p + 10x_p = 2A + 10(At + B) = 50t \implies A = 5, B = -1$$

$$x(t) = e^{-t}\left(c_1 \cos 3t + c_2 \sin 3t\right) + 5t - 1$$

$$x(0) = c_1 - 1 = 1 \implies c_1 = 2$$

$$x(t) = e^{-t}\left(2 \cos 3t + c_2 \sin 3t\right) + 5t - 1$$

$$\dot{x}(t) = e^{-t}\left(-6 \sin 3t + 3c_2 \cos 3t\right) - e^{-t}\left(2 \cos 3t + c_2 \sin 3t\right) + 5$$

$$\dot{x}(0) = 3c_2 - 2 + 5 = -3 \implies c_2 = -2$$

$$x(t) = \underbrace{2e^{-t}\left(\cos 3t - \sin 3t\right)}_{\text{transient}} + 5t - 1 \qquad \text{there is no steady state solution}$$

(b) i. $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{1}} = \sqrt{10} = \omega_f$

ii. $x_p = At \cos \sqrt{10}t + Bt \sin \sqrt{10}t$

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6. [2360/112024 (20 pts)] Use Laplace transforms to solve $3y - y' = -18t,\ y(0) = 5$.

SOLUTION:

$$\mathscr{L}\{y' - 3y = 18t\}$$

$$sY(s) - y(0) - 3Y(s) = \frac{18}{s^2}$$

$$(s-3)Y(s) = \frac{18}{s^2} + 5$$

$$Y(s) = \frac{18}{s^2(s-3)} + \frac{5}{s-3}$$

$$y(t) = \mathscr{L}^{-1}\left\{\frac{18}{s^2(s-3)} + \frac{5}{s-3}\right\}$$

$$\frac{18}{s^2(s-3)} = \frac{As+B}{s^2} + \frac{C}{s-3}$$

$$18 = (As+B)(s-3) + Cs^2$$

$$s = 3 : 18 = 9C \implies C = 2$$

$$s = 0 : 18 = B(-3) \implies B = -6$$

$$s = 1 : 18 = (A-6)(-2) + 2 \implies A = -2$$

$$y(t) = \mathscr{L}^{-1}\left\{\frac{-2s-6}{s^2} + \frac{2}{s-3} + \frac{5}{s-3}\right\}$$

$$= \mathscr{L}^{-1}\left\{-\frac{2}{s} - \frac{6}{s^2} + \frac{7}{s-3}\right\}$$

$$= -2 - 6t + 7e^{3t}$$

■