- This exam is worth 100 points and has 6 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"$\times$ 11" crib sheet with writing on one side.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting. Remote students cannot interact with anyone except the proctor during the exam.

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0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/112024 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) If the equation of motion for a certain undamped mass/spring harmonic oscillator having a spring(restoring) constant of 2 N/m is $x(t) = 10\cos\left(\frac{t}{2}\right) - 5\sin 2t$, then the mass is 8 kg.

   (b) The differential equation $y''' - t = 0$ can be written as a system of three first order equations in the form $\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & t \end{bmatrix} \vec{\mathbf{x}}$.

   (c) $\ddot{x} - tx = 0$ describes a conservative system.

   (d) The solution space of $y^{(4)} + 8y'' + 16y = 0$ is span $\{\cos 2t, \sin 2t, t\cos 2t, t\sin 2t\}$.

   (e) Let $L(\vec{\mathbf{y}}) = f(t)$ where L represents a linear differential operator with constant coefficients. If the characteristic equation of $L(\vec{\mathbf{y}}) = 0$ is $r^3(r-1)^2 = 0$ and $f(t) = 3t\left(e^t - 2t^2\right)$, then the guess for the particular solution when using the method of undetermined coefficients is $y_p = \left(At^3 + Bt^2\right)e^t + Ct^3 + Dt^2 + Et + F$.

2. [2360/112024 (12 pts)] Consider the ordinary differential equation $y''' = 9y'$.

   (a) (6 pts) Determine the general solution of the differential equation.

   (b) (6 pts) Verify that the solutions found in part (a) are linearly independent.

3. [2360/112024 (12 pts)] You received a toy spring with a 0.25 kg mass attached to it as a recent birthday present. Recognizing the toy as a harmonic oscillator and being mathematically inclined you, of course, have made some measurements:

   - 8 N are required to stretch the spring 2 m
   - when moving at 6 m/s, the mass experiences a damping force of 12 N

   (a) (6 pts) One day, while playing with the toy, you pull the mass $\frac{1}{2}$ m to the right of its equilibrium position and then push it to the left at 3 m/sec. Write, but **do not solve**, the initial value problem governing the displacement, $x(t)$, of the oscillator if an external force of te^{-t} is acting on it.

   (b) (6 pts) Without finding $x(t)$ in part (a), answer the following.

      i. (3 pts) Providing mathematical justification, determine whether the oscillator is underdamped, overdamped or critically damped.

      ii. (3 pts) Will you get the thrill of watching the mass pass through the equilibrium position twice? Why or why not?

4. [2360/112024 (20 pts)] Consider the differential equation $x^2z'' - xz' = x^2\ln x,\ x > 0$.

   (a) (8 pts) Assuming solutions of the form $z = x^r$, find a basis for the solution space of the associated homogeneous equation.

   (b) (12 pts) Find the general solution of the original, nonhomogeneous equation.

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE BELOW/ON REVERSE

5. [2360/112024 (26 pts)] Consider the harmonic oscillator described by the initial value problem

$$\ddot{x} + 2\dot{x} + 10x = 50t, \quad x(0) = 1, \quad \dot{x}(0) = -3$$

(a) (20 pts) Find the transient and steady state solutions, if any exist.

(b) (6 pts) Now suppose the damping is removed and the oscillator is given a driving force of the form $F_0 \cos \omega_f t$.

i. (3 pts) Find the frequency, ω_f, that will put the system into pure resonance.

ii. (3 pts) If you were to solve the differential equation modeling this new situation with the ω_f from part (i) (do not do so), what would be the **form** of the particular solution?

6. [2360/112024 (20 pts)] Use Laplace transforms to solve $3y - y' = -18t,\ y(0) = 5$.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\mathrm{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\mathrm{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
