BCS Theory - Preliminaries

- Turns out that (in cases we understand well, anyway) attractive electron-electron interactions are important for superconductivity. Where can this come from? Phonons!

(In reality, there is both an attractive part of the interaction due to phonons and a repulsive part from Coulomb's interaction... for simplicity we will neglect Coulomb.)

Phonons \rightarrow \text{Attractive Interaction}

- Very simple picture of phonons (Einstein model)
  \[ \omega \sim \sqrt{\frac{K}{M}} = \omega_D, \text{ typical frequency} \]
  \[ \frac{\hbar \omega}{K_B} \sim 10^2 \text{K} \approx 1000 \text{K} \]
Suppose electron comes along, virtually excites phonon...

This phonon needs to get rid of its energy... gives it to another electron e₂.

Because the phonon needs to give up its energy, it tends to pull the second electron nearby to do so → attractive interaction.
• This picture only makes sense for electrons and holes with energies in a shell of width ~ twice about the Fermi surface.

• Crude Hamiltonian describing these: restrict E_k, E_k', E_k'' etc. all lie within shell.

\[
H_{int} = - \frac{U_0}{V} \sum \sum \sum_{\sigma, \sigma'} \psi^+_\sigma (k+q) \psi^+_\sigma' (k-q) \psi_\sigma (k') \psi_\sigma' (k'')
\]

→ Corresponds to attractive density-density interaction.
"Cooper Problem"

- Consider a "frozen" filled Fermi sea with 2 extra electrons outside $k_F$. \(\Rightarrow\) Solve 2 electron problem, when 2 electrons can only occupy states with \(|\mathbf{k}| > k_F\).

- Assume spins form singlet (\(\Rightarrow\) spatial part of wavefunction is symmetric, greatest energy gain from attractive interaction).
  Also assume zero net momentum.
  \(\Rightarrow\) General wavefunction is:

\[
\Psi(r_1, r_2) = \sum_{\mathbf{k}, \mathbf{l}} g(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}_1} e^{-i\mathbf{k} \cdot \mathbf{r}_2}
\]

\[
\Psi(r_1, r_2) = \Psi(r_2, r_1) \Rightarrow g(\mathbf{k}) = g(-\mathbf{k})
\]

- Effect of interaction on $\Psi_0$ state (consider $\Psi_{k_1}^\dagger \Psi_{-k_2}^\dagger |\Psi_S\rangle$)
\[ H_{\text{int}} | \vec{F} \rangle = - \frac{U_0}{V} \sum \sum' \psi_\uparrow^+(k + q) \psi_\downarrow^+(k' - q) \psi_\uparrow^-(k) \psi_\downarrow^-(k') | FS \rangle \]

Two terms

\[ \sigma = \uparrow, \sigma' = \downarrow, \vec{k} = \vec{k}, \vec{k}' = -\vec{k} \]

\[ - \frac{U_0}{V} \sum \psi_\uparrow^+(k + q) \psi_\downarrow^+(k' - q) | FS \rangle \]

\[ - \frac{U_0}{V} \sum \psi_\uparrow^+(k - q) \psi_\downarrow^+(k + q) | FS \rangle \]

\[ = - \frac{2U_0}{V} \sum \psi_\uparrow^+(q) \psi_\downarrow^+(q) | FS \rangle \]

\[ \Rightarrow \text{Write: } H_{\text{int}} = - \sigma \sigma' \sum' | k \rangle \langle k'| - \frac{2U_0}{V} \sum_{k,k'} \langle k | k' \rangle \langle k' | k \rangle \]

\[ k_f < k, k' < k_f + 5k \]
Z-body Hamiltonian:

\[ H = 2 \sum_{k} \varepsilon(k) |k\rangle \langle k| - \frac{2U_0}{V} \sum_{k,k'} |k\rangle \langle k'| \langle k'| \langle k| \]  

(k \leq k_f, k < k_f + 5k)  

Wavefunction: \[ |\Psi\rangle = \sum_{n} c_n |n\rangle \]  

H|\Psi\rangle = E|\Psi\rangle \rightarrow \text{Schrödinger Equation}  

\[ 2\varepsilon(k) g(k) - \frac{2U_0}{V} \sum_{k'} g(k') = E g(k) \]  

(k \leq k_f, k < k_f + 5k)  

\[ \Rightarrow g(k) = \frac{2U_0}{V} \sum_{k'} \frac{g(k')}{\varepsilon(k') - \varepsilon(k)} \]  

This step only safe if \( E < 2\varepsilon_f \); otherwise can have divergences.
\[ 1 = \frac{2U_0}{V} \sum_{\xi} \frac{1}{Z\xi(E) - E} \]

\[ = \int \frac{2U_0}{V} V \int_{E_F}^{E_F + \hbar \omega_D} dE \frac{D(E)}{Z\xi_{E_F} - E} \]

\[ \propto \frac{2U_0}{V} D(E_F) \int_{E_F}^{E_F + \hbar \omega_D} \frac{dE}{Z\xi_{E_F} - E} \]

Assume \( U_0 D(E_F) \ll 1 \) \( \rightarrow \) "weak coupling" limit.

\[ \Rightarrow \frac{1}{2U_0 D(E_F)} = \frac{1}{Z\xi} \left| \frac{2Z\xi_{E} - E + 2\hbar \omega_D}{Z\xi_{E} - E} \right| \]

\[ \propto \frac{1}{Z\xi} \left| \frac{2\hbar \omega_D}{Z\xi_{E} - E} \right| \]
* Experiment: Solve for $E$...

$$E \approx 2\epsilon_F - 2\omega_0 e^{-1/v_0 D(\epsilon_F)}$$

* Found a bound state with energy below $2\epsilon_F$...
  Suggests Fermi sea is unstable!

* $E$ is non-analytic in $V_0$ ... would never find this by doing perturbation theory in $V_0$.

* Bound state is a boson (Cooper pair). We might guess that such bosons want to condense... leading to superfluidity as in dilute Bose gas.