1. Note that I ignore spin throughout this solution – to put it back in, you just need to multiply various equations by a factor of 2. This doesn’t affect the final result for the collisionless Boltzmann equation.

First, let’s think about the geometry of the problem. Phase space is six-dimensional now, and I do not know how to visualize anything! But we can describe the geometry in words and equations. We’re interested in the flow of electrons in and out of a small six-dimensional hypercube of phase space near the point \((r, k)\). The corners of this region are \((r, k), (r + dx, x, k), (r, k + dk_x y), etc\). So the total phase space volume of this region is \(d^3r d^3k\), and the number of electrons inside (ignoring spin) is \(dN = g(r, k, t)d^3r d^3k/(2\pi)^3\).

Now, in the 1d case, our region of phase was bounded by four one-dimensional surfaces (i.e., lines). It was natural to think of these lines in two pairs. One pair had \(r\) as the normal direction, and the other pair had \(k\) as the normal direction. Our 6d region is bounded by twelve 5d hypersurfaces. These break into six pairs, and each pair has one of the six phase space coordinates as its normal direction. For example, let’s consider the \(r_x\)-pair, the pair of hypersurfaces with \(r_x\) as the normal direction. Let \((r', k')\) be the coordinate of a point on one of these hypersurfaces. For one of the surfaces (call this the “bottom” surface), \(r'_x = r_x\), and the other phase space coordinates are free to vary: \(r_y \leq r'_y \leq r_y + dy, k_z \leq k'_z \leq k_x + dk_x\), and similarly for \(r'_y, k'_y, k'_z\). The other \(r_x\)-surface (the “top” surface) has \(r'_x = r_x + dr_x\), and again the other 5 coordinates are free to vary.

Let’s calculate the flow of particles in through the two \(r_x\)-surfaces. The flow in through the bottom surface is:

\[
\left[ g(r, k, t) \frac{dr_x}{dt} (r, k, t) dt \right] dr_y dr_z \frac{d^3k}{(2\pi)^3} \tag{1}
\]

The factor outside the square brackets is the 5d volume element of the hypersurface (together with our conventional factors of 2\(\pi\)). Next, the flow in through the top surface is

\[
- \left[ g(r' + dr_x, x, k', t) \frac{dr_x}{dt} (r + dr_x, x, k, t) dt \right] dr_y dr_z \frac{d^3k}{(2\pi)^3} = - \left[ g(r + dr_x, x, k, t) \frac{dr_x}{dt} (r, k, t) dt \right] dr_y dr_z \frac{d^3k}{(2\pi)^3}, \tag{2}
\]

where we used the fact that \(dr/dt = v(k)\), which is independent of \(r\). Adding these together and expanding in \(dr_x\) gives the total through number in through the \(r_x\)-pair:

\[
\left[ - \frac{\partial g}{\partial r_x} \frac{dr_x}{dt} \right] d^3r \frac{d^3k}{(2\pi)^3} dt \tag{3}
\]

Looking at the form of this result, it’s easy to write down the flow in through the \(r_y\)- and \(r_z\)-pairs, and the total flow through all the \(r\)-pairs is then:

\[
\left[ - \frac{\partial g}{\partial r} \cdot \frac{dr}{dt} \right] d^3r \frac{d^3k}{(2\pi)^3} dt. \tag{4}
\]

So far this is looking like a completely straightforward generalization of our 1d calculation in class. But we will see that it’s a bit different in 3d for the \(k\)-pairs. Let’s consider “bottom” \((k'_x = k_x)\) and “top” \((k'_x = k_x + dk_x)\) surfaces belong to the \(k_x\)-pair. The flow in through the bottom surface is

\[
\left[ g(r, k, t) \frac{dk_x}{dt} (r, k, t) dt \right] d^3r \frac{dk_y dk_z}{(2\pi)^2}, \tag{5}
\]

And, the flow in through the top surface is

\[
- \left[ g(r, k + dk_x x, t) \frac{dk_x}{dt} (r, k + dk_x x, t) dt \right] d^3r \frac{dk_y dk_z}{(2\pi)^2}. \tag{6}
\]

In this case, \(dk/\) depends on \(k\) through

\[
\frac{dk}{dt} = - \frac{e}{\hbar} E + \frac{e}{\hbar c} v(k) \times B. \tag{7}
\]
So we can’t just naively drop the $dk_x$-dependence in $dk_x/dt$. Instead, when we add the contributions from the
two surfaces together, and expand in $dk_x$, we get

$$-\left[ \frac{\partial g}{\partial k_x} \frac{dk_x}{dt} + g \frac{\partial}{\partial k_x} \frac{dk_x}{dt} \right] d^3r \frac{d^3k}{(2\pi)^3} dt \quad (8)$$

Adding all the contributions from $k$-pairs together, and defining $\nabla_k = \partial/\partial k$, we then get

$$-\left[ \frac{\partial g}{\partial k} \frac{dk}{dt} + e/\hbar g \nabla_k \cdot (v(k) \times B) \right] d^3r \frac{d^3k}{(2\pi)^3} dt \quad (9)$$

Now, recalling that

$$v(k) = \frac{1}{\hbar} \frac{1}{\partial \epsilon(k)}$$

we have

$$\nabla_k \cdot (v(k) \times B) = \frac{1}{\hbar} \epsilon_{ijk} \frac{\partial \epsilon(k)}{\partial k_i} \partial_k B_k = \frac{1}{\hbar} \epsilon_{ijk} \frac{\partial^2 \epsilon(k)}{\partial k_i \partial k_j} B_k = 0. \quad (11)$$

This vanishes because $\epsilon_{ijk} = -\epsilon_{jik}$, while the mixed partial derivative is symmetric:

$$\frac{\partial^2 \epsilon(k)}{\partial k_i \partial k_j} = -\frac{\partial^2 \epsilon(k)}{\partial k_j \partial k_i} \quad (12)$$

So the total contribution from the $k$-pairs is then just

$$-\left[ \frac{\partial g}{\partial k} \frac{dk}{dt} \right] d^3r \frac{d^3k}{(2\pi)^3} dt, \quad (13)$$

with no extra term, just like in the 1d case.

We can now follow the 1d derivation from class to write down the Boltzmann equation. The total number of
electrons going into the region is

$$\left[ \frac{\partial g}{\partial t} \right] d^3r \frac{d^3k}{(2\pi)^3} dt = -\left[ \frac{\partial g}{\partial r} \frac{dr}{dt} + \frac{\partial g}{\partial k} \frac{dk}{dt} \right] d^3r \frac{d^3k}{(2\pi)^3} dt, \quad (14)$$

so we have

$$\frac{\partial g}{\partial t} = -\frac{\partial g}{\partial r} \frac{dr}{dt} - \frac{\partial g}{\partial k} \frac{dk}{dt}. \quad (15)$$

2. First let’s simplify the expression a little bit:

$$g(k) - g^{(0)}(k) = \tau \left[ - f'(\epsilon_k) \right] v_k \cdot \left[ \frac{\epsilon_k - \mu}{T} \right] (-\nabla T) \quad (16)$$

$$= \tau \left[ - f'(\epsilon_k) \right] \frac{1}{\hbar} \frac{1}{\partial \epsilon_k} \partial_k \left[ \frac{\epsilon_k - \mu}{T} \right] (-\nabla T) \quad (17)$$

$$= \tau \frac{\partial g^{(0)}(k)}{\partial k} \cdot \left[ \frac{\epsilon_k - \mu}{T} \right] (-\nabla T). \quad (18)$$

In the last line we used the chain rule, and $g^{(0)}(k) = f(\epsilon_k)$. We can now follow the argument given in class and
write this schematically as

$$g(k) = g^{(0)}(k) + \frac{\partial g^{(0)}(k)}{\partial k} \Delta k, \quad (19)$$
where
\[ \Delta k \approx \frac{\tau}{\hbar} \frac{|\epsilon_k - \mu|}{T} |\nabla T|, \quad (20) \]

There is one tricky thing – this expression is still a function of \( k \). But we can recall that the only states that will contribute substantially to transport are those right on the Fermi surface, or thermally excited states near the Fermi surface. So, at most, \( |\epsilon_k - \mu| \) will be on the order of \( k_B T \). Then we have
\[ \Delta k \approx \frac{k_B \tau}{\hbar} |\nabla T| \approx 0.01 \text{ cm}^{-1}, \quad (21) \]

for \( \tau = 10^{-14} \text{ s} \) and \( |\nabla T| = 10 \text{ K/cm} \). This gives
\[ \frac{\Delta k}{k_F} \approx 10^{-10} \quad (22) \]

for \( k_F = 10^8 \text{ cm}^{-1} \). This is really small, so the shift in the equilibrium distribution is indeed very small for reasonable parameters.

In fact, although it was not part of the problem, it’s also a good idea to compare \( \Delta k \) to \((\Delta k)_T\), which I define to be the width of the thermal distribution in momentum space. That is, it’s the distance in \( k \)-space from the Fermi surface to a typical excited state at a given temperature. We have
\[ (\Delta k)_T = \frac{k_B T}{\hbar v_F}, \quad (23) \]

where \( v_F = \hbar k_F/m \) is the Fermi velocity. For \( T = 1 \text{ K} \) and \( v_F = 10^8 \text{ cm/s} \), and using the same parameters to estimate \( \Delta k \), we have
\[ \frac{\Delta k}{(\Delta k)_T} \approx 10^{-5} \ll 1. \quad (24) \]

Therefore the shift of the equilibrium distribution is very small even compared to its thermal width near the Fermi surface! One further point worth noting is that \( |\nabla T| = 10 \text{ K/cm} \) is a very large thermal gradient at low temperatures – this means that the ends of a 1 cm long sample would be 10 K apart in temperature. In typical measurements of thermal transport, one wants to be able to say the sample is held at a constant temperature, so this temperature difference from end to end must be much less than average temperature. For smaller \( |\nabla T| \), it is of course an even better approximation to linearize in the thermal gradient.

3. We start with the expressions relating the electrical and thermal currents to the applied field and thermal gradient, which thus define the transport coefficients:
\[ J_i = (L_{EE})_{ij} \mathcal{E}_j + (L_{ET})_{ij} (-\nabla T)_j, \quad (25) \]
\[ J_{ij}^Q = (L_{TE})_{ij} \mathcal{E}_j + (L_{TT})_{ij} (-\nabla T)_j. \quad (26) \]

The sum over the repeated index \( j \) is implied. The strategy is to plug the given expression for \( g(k) \) into the general expressions for the currents, and then read off the coefficients of \( \mathcal{E} \) and \( \nabla T \).

For the electrical current we have
\[ J_i = \int \frac{d^3 k}{4\pi^3} (-e v^i_k) g(k) \]
\[ = \int \frac{d^3 k}{4\pi^3} (-e v^i_k) \left[ g(k) - g^{(0)}(k) \right] \]
\[ = -e \tau \int \frac{d^3 k}{4\pi^3} \left[ -f'(\epsilon_k) \right] v^i_k v^j_k \left[ \left( \frac{\epsilon_k - \mu}{T} \right) (-\nabla T)_j - e \mathcal{E}_j \right] \]
\[ = \left[ -\frac{e \tau}{T} \int \frac{d^3 k}{4\pi^3} \left[ -f'(\epsilon_k) \right] (\epsilon_k - \mu) v^i_k v^j_k \right] (-\nabla T)_j + \left[ e^2 \tau \int \frac{d^3 k}{4\pi^3} \left[ -f'(\epsilon_k) \right] v^i_k v^j_k \right] \mathcal{E}_j \]
This implies
\[(L_{EE})_{ij} = e^2 T \int \frac{d^3 k}{4\pi^3} [-f'(\epsilon_k)] v_i^k v_j^k, \quad (31)\]
and
\[(L_{ET})_{ij} = -e T \int \frac{d^3 k}{4\pi^3} [-f'(\epsilon_k)] (\epsilon_k - \mu) v_i^k v_j^k. \quad (32)\]

Following the same procedure, but for the thermal current \(j_i^Q\), we find
\[(L_{TE})_{ij} = -e T \int \frac{d^3 k}{4\pi^3} [-f'(\epsilon_k)] (\epsilon_k - \mu) v_i^k v_j^k, \quad (33)\]
and
\[(L_{TT})_{ij} = \frac{T}{T} \int \frac{d^3 k}{4\pi^3} [-f'(\epsilon_k)] (\epsilon_k - \mu)^2 v_i^k v_j^k. \quad (34)\]

By comparing the two expressions given above, it immediately follows that
\[(L_{TE})_{ij} = T(L_{ET})_{ij}. \quad (35)\]

We now evaluate \((L_{TT})_{ij}\):
\[(L_{TT})_{ij} = \frac{T}{T} \int \frac{d^3 k}{4\pi^3} [-f'(\epsilon_k)] (\epsilon_k - \mu)^2 k_i k_j \quad (36)\]
\[= \frac{T}{3T} \int \frac{d^3 k}{4\pi^3} [-f'(\epsilon_k)] (\epsilon_k - \mu)^2 k^2 \quad (37)\]
\[= \frac{2T}{3mT} \int \frac{d^3 k}{4\pi^3} [-f'(\epsilon_k)] (\epsilon_k - \mu)^2, \quad (38)\]

where in the second line we used the trick discussed in class, where we may replace \(k_i k_j \rightarrow (1/3) k^2 \delta_{ij}\) when the rest of the integrand is spherically symmetric, and in the third line we used \(\epsilon_k = \hbar^2 k^2/(2m)\). Since the integrand depends only on \(\epsilon_k\), we may replace it with an integral over energy using the density of states. We also write \((L_{TT})_{ij} = \delta_{ij} L_{TT}\), and we have
\[L_{TT} = \frac{2T}{3mT} \int_{-\infty}^{\infty} \epsilon \rho(\epsilon) [-f'(\epsilon)](\epsilon - \mu)^2, \quad (39)\]

where \(\rho(\epsilon)\) is the density of states. We can evaluate this integral using the Sommerfeld expansion. By following the derivation in Appendix C of Ashcroft and Mermin, we have
\[\int_{-\infty}^{\infty} \epsilon \rho(\epsilon) [-f'(\epsilon)] K(\epsilon) = K(\mu) + \frac{\pi^2}{6} k_B T^2 K''(\epsilon_F) + O(T^4), \quad (40)\]

where \(K(\epsilon)\) is a general function of energy, which must be well-behaved near \(\epsilon = \mu\). Note that we have replaced \(K''(\mu) \rightarrow K''(\epsilon_F)\) in the second term – this only introduces errors of order \(T^4\), which we are neglecting anyway. In the integral for \(L_{TT}\), we have \(K(\epsilon) = \epsilon \rho(\epsilon)(\epsilon - \mu)^2\). Then \(K(\mu) = 0\), and after some algebra it can be shown that
\[K''(\epsilon_F) = 3n. \quad (41)\]

Putting everything together, we have the result
\[L_{TT} = \frac{\pi^2 nT}{3} k_B^2 T. \quad (42)\]
Now, using the same tricks as above, we evaluate \((L_{ET})_{ij}\):

\[
(L_{ET})_{ij} = \frac{-e\tau}{4\pi^3} \iint d^3k \left[ -f'(\epsilon_k) \right] (\epsilon_k - \mu) v_k^i v_k^j
\]  

(43)

\[
= -\delta_{ij} \frac{e\tau}{4\pi^3} \iint d^3k \left[ -f'(\epsilon_k) \right] (\epsilon_k - \mu) k^2
\]  

(44)

\[
= -\delta_{ij} \frac{2e\tau}{3mT} \iint d^3k \left[ -f'(\epsilon_k) \right] (\epsilon_k - \mu) \epsilon_k.
\]  

(45)

Again we write \((L_{ET})_{ij} = \delta_{ij} L_{ET}\), and go to an integral over energy:

\[
L_{ET} = -\frac{2e\tau}{3mT} \int_{-\infty}^{\infty} d\epsilon D(\epsilon) \left[ -f'(\epsilon) \right] \epsilon (\epsilon - \mu).
\]  

(46)

Now we have \(K(\epsilon) = \epsilon D(\epsilon) (\epsilon - \mu)\). Again \(K(\mu) = 0\), and now

\[
K''(\epsilon_F) = \frac{9}{2} \frac{n}{\epsilon_F^2}.
\]  

(47)

Putting everything together, we have,

\[
L_{ET} = \frac{n e^2 \tau}{m} \left[ -\frac{\pi^2 k_B^2 T}{2 \epsilon_F^2} \right].
\]  

(48)

4. (a) Since the applied fields do not depend on time or space, we should expect a homogeneous, steady-state solution. So we will assume

\[
\frac{\partial g}{\partial t} = \frac{\partial g}{\partial \epsilon} = 0.
\]  

(49)

The Boltzmann equation is then

\[
\dot{k} \cdot \frac{\partial g}{\partial k} = -\frac{1}{\tau} (g(k) - g^{(0)}(k)),
\]  

(50)

where \(g^{(0)}(k) = f(\epsilon_k)\) is the equilibrium distribution, and

\[
\dot{k} = -\frac{e}{\hbar} [E + \frac{1}{c} (v_k \times B)].
\]  

(51)

Linearizing in the electric field, the left hand side of the Boltzmann equation becomes

\[
\dot{k} \cdot \frac{\partial g}{\partial k} = -\frac{e}{\hbar} E \cdot \frac{\partial f(\epsilon_k)}{\partial k} - \frac{e}{\hbar c} (v_k \times B) \cdot \frac{\partial g}{\partial k}
\]  

(52)

\[
= -e f'(\epsilon_k) v_k \cdot E - \frac{e}{\hbar c} (v_k \times B) \cdot \frac{\partial g}{\partial k},
\]  

(53)

where in the second line we used the chain rule and the fact that \(v_k = (1/\hbar)(\partial \epsilon_k/\partial k)\). Note that in the second term, involving the magnetic field, we cannot replace \(g\) with \(g^{(0)}\) – this is because we want to keep terms of first order in \(E\), but the only \(E\)-dependence in this term will come from \((\partial g/\partial k)\).

Now our strategy is to simply plug in the trial solution

\[
g(k) - g^{(0)}(k) = [-f'(\epsilon_k)] v_k \cdot A,
\]  

(54)

given in the statement of the problem. Remember that we assume \(A\) does not depend on \(k\). \(A\) will depend on \(E\) and \(B\). Since nothing happens if we don’t apply the electric field, we expect \(A = 0\) for \(E = 0\). Since we are linearizing in \(E\), we should then assume that \(A\) is a linear function of \(E\). To plug this in, we have to deal with two terms in the Boltzmann equation. The right-hand side is simply

\[
-\frac{1}{\tau} (g(k) - g^{(0)}(k)) = -\frac{1}{\tau} [-f'(\epsilon_k)] v_k \cdot A.
\]  

(55)
Next we have to deal with the magnetic field term on the left-hand side

\[-\frac{e}{\hbar c} (v_k \times B) \cdot \frac{\partial g}{\partial k} = -\frac{e}{\hbar c} (v_k \times B) \cdot \left[ \frac{\partial f(\epsilon_k)}{\partial k} + \frac{\partial}{\partial k} \left[ -f'(\epsilon_k) v_k \cdot A \right] \right] \tag{56} \]

\[= -\frac{e}{c} (v_k \times B) \cdot \left[ f'(\epsilon_k) v_k - f''(\epsilon_k) v_k (v_k \cdot A) + \frac{1}{\hbar} \frac{\partial}{\partial k} \left[ f' v_k A_i \right] \right]. \tag{57} \]

The first two terms vanish, because \((v_k \times B) \cdot v_k = 0\), and we have

\[-\frac{e}{\hbar c} (v_k \times B) \cdot \frac{\partial g}{\partial k} = -\frac{e}{\hbar c} (v_k \times B) \cdot \left[ f'(\epsilon_k) v_k \right] \cdot \left[ -f'(\epsilon_k) v_k \cdot (B \times A) \right]. \tag{58} \]

Now,

\[\frac{\partial v_i^k}{\partial k_j} = \hbar \frac{\partial k_i}{\partial k_j} = \delta_{ij} \frac{\hbar}{m}. \tag{59} \]

Putting this in, we have

\[-\frac{e}{\hbar c} (v_k \times B) \cdot \frac{\partial g}{\partial k} = -\frac{e}{mc} [f'(\epsilon_k)] (v_k \times B) \cdot A = -\frac{e}{mc} [f'(\epsilon_k)] v_k \cdot (B \times A), \tag{60} \]

where in the second equality we used the cyclic property of the triple product.

Putting these results together, we have for the Boltzmann equation

\[f'(\epsilon_k) v_k \cdot \left[ -eE + \frac{e}{mc} B \times A \right] = f'(\epsilon_k) v_k \cdot \left[ \frac{1}{\tau} A \right]. \tag{61} \]

Now, this equation should hold for all \(k\), but \(A\) does not depend on \(k\). The only way for this to be true is if

\[-eE + \frac{e}{mc} B \times A = \frac{1}{\tau} A. \tag{62} \]

This is an algebraic equation we can solve for \(A\)! We look for a solution of the form

\[A = c_1 E + c_2 (E \times B). \tag{63} \]

Because we have assumed \(E\) and \(B\) are perpendicular, then \(E\) and \(E \times B\) are also perpendicular. Note that there is no term in \(A\) proportional to \(B\), since we should have \(A = 0\) for \(E = 0\). If we plug this form into Eq. (62), we find

\[E \left[ -e + \frac{eB^2}{mc^2} c_2 - \frac{1}{\tau} c_1 \right] + (E \times B) \left[ -\frac{e}{mc} c_2 - \frac{1}{\tau} c_1 \right] = 0, \tag{64} \]

where we used the fact that \(B \times (E \times B) = B^2 E\). Because \(E\) and \((E \times B)\) are perpendicular we must set their coefficients separately for zero, and we obtain the following system of linear equations for \(c_1\) and \(c_2\): \[\frac{1}{\tau} c_1 - \frac{eB^2}{mc^2} c_2 = -e \tag{65} \]

\[\frac{e}{mc} c_1 + \frac{1}{\tau} c_2 = 0. \tag{66} \]

Solving these we have

\[c_1 = \frac{-e\tau}{1 + (\omega_c \tau)^2} \tag{67} \]

\[c_2 = \frac{\tau^2 e^2}{mc} \frac{1}{1 + (\omega_c \tau)^2}. \tag{68} \]

where we introduced the cyclotron frequency

\[\omega_c = \frac{eB}{mc}. \tag{69} \]
We have for $A$:

$$A = \frac{-e\tau}{1 + (\omega_c \tau)^2} \left[ E - (\omega_c \tau) E \times \hat{B} \right],$$  \hspace{1cm} (70)$$

where $\hat{B} = B / |B|$ is the unit vector along the direction of $B$.

We can now plug our solution into the expression for the electric current density

$$J = \frac{e^2 \tau}{1 + (\omega_c \tau)^2} \int \frac{d^3 k}{4\pi} \left[ - f'(\epsilon_k) \right] v_k \left( v_k \cdot [E - (\omega_c \tau) E \times \hat{B}] \right).$$  \hspace{1cm} (71)$$

Now we assume that $E = Ex$ and $B = Bz$. Then, using $E \times \hat{B} = -Ey$, we have

$$J_x = \frac{e^2 \tau}{1 + (\omega_c \tau)^2} \int \frac{d^3 k}{4\pi} \left[ - f'(\epsilon_k) \right] v_k^x v_k^y E + (\omega_c \tau) E v_k^y$$

$$= \frac{e^2 \tau}{1 + (\omega_c \tau)^2} \int \frac{d^3 k}{4\pi} \left[ - f'(\epsilon_k) \right] [v_k^y]^2.$$  \hspace{1cm} (72)$$

The second term dropped out because the integrand was odd in $k_x$ and $k_y$. We then have

$$\sigma_{xx} = \frac{e^2 \tau}{1 + (\omega_c \tau)^2} \int \frac{d^3 k}{4\pi^3} \left[ - f'(\epsilon_k) \right] [v_k^y]^2$$

$$= \frac{\hbar^2}{3m^2} \frac{e^2 \tau}{1 + (\omega_c \tau)^2} \int \frac{d^3 k}{4\pi^3} \left[ - f'(\epsilon_k) \right] k^2$$

$$= \frac{2}{3m} \frac{e^2 \tau}{1 + (\omega_c \tau)^2} \int \frac{d^3 k}{4\pi^3} \left[ - f'(\epsilon_k) \right] \epsilon_k.$$  \hspace{1cm} (73)$$

This integral is exactly what we evaluated in class to find the electrical conductivity for $B = 0$; we have the result

$$\sigma_{xx} = \frac{1}{1 + (\omega_c \tau)^2} \frac{2e^2 \tau}{m}.$$  \hspace{1cm} (74)$$

Now, in order to find $\sigma_{yx}$, we calculate the $y$-component of the current density:

$$J_y = \frac{e^2 \tau}{1 + (\omega_c \tau)^2} \int \frac{d^3 k}{4\pi^3} \left[ - f'(\epsilon_k) \right] v_k^y \left( v_k^x E + (\omega_c \tau) E v_k^x \right)$$

$$= \frac{e^2 \tau}{1 + (\omega_c \tau)^2} \int \frac{d^3 k}{4\pi^3} \left[ - f'(\epsilon_k) \right] [v_k^y]^2.$$  \hspace{1cm} (75)$$

We already considered this integral above (it makes no difference that we have $v_k^y$ instead of $v_k^x$), so we can immediately write down the result

$$\sigma_{yx} = \frac{\omega_c \tau}{1 + (\omega_c \tau)^2} \frac{2e^2 \tau}{m}.$$  \hspace{1cm} (76)$$

(b) As above, we look for a steady state solution, $\partial g / \partial t = 0$. However, in this case $\partial g / \partial r \neq 0$. As we did in class (see the “thermal transport” set of lecture notes), we linearize in $\nabla T$ to find

$$\frac{\partial g}{\partial r} \approx \left[ - f'(\epsilon_k) \right] \left( \frac{\epsilon_k - \mu}{T} \right) \nabla T.$$  \hspace{1cm} (77)$$

Then the Boltzmann equation becomes

$$[- f'(\epsilon_k)] \left( \frac{\epsilon_k - \mu}{T} \right) v_k \cdot \nabla T - \frac{e}{\hbar c} (v_k \times B) \cdot \frac{\partial g}{\partial k} = -\frac{1}{\tau} [g(r, k) - g^{(0)}(r, k)].$$  \hspace{1cm} (78)$$

Note that, here,

$$g^{(0)}(r, k) = f(\epsilon_k; T(r)),$$  \hspace{1cm} (79)$$
that is, \( g^{(0)} \) is the local equilibrium distribution, given by the Fermi function evaluated using the local temperature \( T(r) \).

We now plug in the trial solution

\[
g(r,k) - g^{(0)}(r,k) = [-f'(\epsilon_k)]v_k \cdot A(\epsilon_k),
\]

where \( A \) is now a function of \( \epsilon_k \). Note that the right-hand side of this equation is independent of position. As before, the only term that is not straightforward to deal with is the magnetic field term, but the procedure for that is just as before. The only difference is that now there will be an additional contribution involving \( \partial A_i / \partial k = \hbar (\partial A_i / \partial \epsilon) v_k \). This contribution will vanish, however, since \( v_k \cdot (v_k \times B) = 0 \). Therefore the Boltzmann equation becomes

\[
f'(\epsilon_k) v_k \cdot \left[ -\left( \frac{\epsilon_k - \mu}{T} \right) \nabla T + \frac{e}{mc} B \times A \right] = f'(\epsilon_k) v_k \cdot \left[ \frac{1}{\tau} A \right].
\]

Again, this leads to an algebraic equation for \( A \):

\[
-\left( \frac{\epsilon_k - \mu}{T} \right) \nabla T + \frac{e}{mc} B \times A = \frac{1}{\tau} A.
\]

This equation is identical to the one we found in part (a), Eq. (62), under the replacement

\[
E \rightarrow \frac{1}{e} \left( \frac{k - \mu}{T} \right) \nabla T.
\]

Therefore we can make this substitution in Eq. (70) to get

\[
A = \frac{-\tau}{1 + (\omega_c \tau)^2} \left( \frac{\epsilon_k - \mu}{T} \right) \left[ \nabla T - (\omega_c \tau) \nabla T \times \hat{B} \right].
\]

Putting our solution into the expression for the thermal current density we have

\[
J_Q = \frac{\tau}{T} \frac{1}{1 + (\omega_c \tau)^2} \int \frac{d^3k}{4\pi^3} \left[ -f'(\epsilon_k) \right] v_k (\epsilon_k - \mu)^2 \left[ v_k \cdot (\nabla T + (\omega_c \tau) \nabla T \times \hat{B}) \right]
\]

Now assuming \( \nabla T = -t_0 \hat{x} \) and \( \hat{B} = B \), we have

\[
J_Q^x = \frac{\tau}{T} \frac{1}{1 + (\omega_c \tau)^2} \int \frac{d^3k}{4\pi^3} \left[ -f'(\epsilon_k) \right] (\epsilon_k - \mu)^2 v_k^y \left[ t_0 v_k^x + t_0 (\omega_c \tau) v_k^y \right]
\]

\[
= t_0 \frac{\tau}{T} \frac{1}{1 + (\omega_c \tau)^2} \int \frac{d^3k}{4\pi^3} \left[ -f'(\epsilon_k) \right] (\epsilon_k - \mu)^2 [v_k^x] \]

From this we see that

\[
\kappa_{xx} = \frac{\tau}{T} \frac{1}{1 + (\omega_c \tau)^2} \int \frac{d^3k}{4\pi^3} \left[ -f'(\epsilon_k) \right] (\epsilon_k - \mu)^2 [v_k^x]^2.
\]

By comparing to the results in problem 2, we can see that

\[
\kappa_{xx} = \frac{1}{1 + (\omega_c \tau)^2} L_{TT}(B = 0) = \frac{\pi^2 n \tau}{3m k_B^2 T},
\]

so we get the same thermal conductivity as for \( B = 0 \), but with an additional prefactor. Now for the \( y \)-component of thermal current. We have

\[
J_Q^y = \frac{\tau}{T} \frac{1}{1 + (\omega_c \tau)^2} \int \frac{d^3k}{4\pi^3} \left[ -f'(\epsilon_k) \right] (\epsilon_k - \mu)^2 v_k^x \left[ t_0 v_k^y + t_0 (\omega_c \tau) v_k^y \right]
\]

\[
= t_0 \frac{\tau}{T} \frac{\omega_c \tau}{1 + (\omega_c \tau)^2} \int \frac{d^3k}{4\pi^3} \left[ -f'(\epsilon_k) \right] (\epsilon_k - \mu)^2 [v_k^x] \]

and therefore

\[
\kappa_{yx} = \frac{\omega_c \tau}{T} \frac{\omega_c \tau}{1 + (\omega_c \tau)^2} \int \frac{d^3k}{4\pi^3} \left[ -f'(\epsilon_k) \right] (\epsilon_k - \mu)^2 [v_k^x]^2
\]

\[
= (\omega_c \tau) \kappa_{xx}.
\]

Yes, the ratios \( \kappa_{xx} / (\sigma_{xx} T) \) and \( \kappa_{yx} / (\sigma_{yx} T) \) both satisfy the Wiedemann-Franz law, independent of magnetic field. This is not surprising, because magnetic field is sort of like another source of elastic scattering – it does no work on electrons during their semiclassical motion and doesn’t change their energy.