1 One dimensional Ising model

(a) Consider a one-dimensional Ising model. It is defined via its partition function

\[ Z = \sum_{\sigma=\pm 1} \exp \left( \frac{J}{T} \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} \right). \]  

This model has a \( \mathbb{Z}_2 \) symmetry.

One dimensional Ising model never orders, because one cannot break symmetry in 1D. Calculate the correlation function of this Ising model defined as

\[ \langle \sigma_n \sigma_m \rangle = \frac{\sum_{\sigma=\pm 1} \sigma_n \sigma_m \exp \left( \frac{J}{T} \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} \right)}{\sum_{\sigma=\pm 1} \exp \left( \frac{J}{T} \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} \right)}. \]

Here \( 1 \leq n < m \leq N \). This correlation function can be computed exactly by employing the following trick. Define \( N - 1 \) variables \( \xi_i = \sigma_i \sigma_{i+1} \) for \( i = 1, 2, \ldots, N - 1 \). Then the partition function can be thought of as summing over variables \( \xi_i \) instead of \( \sigma_i \).

Rewrite the correlation function using the variables \( \xi_i \) (think how to express the product \( \sigma_n \sigma_m \) in terms of these variables) and compute the correlation function. Bring the expression for the correlation function to the form

\[ \langle \sigma_n \sigma_m \rangle = \exp \left( - \frac{m - n}{\ell} \right), \]

where \( \ell \) is referred to as correlation length. Find \( \ell \), and check that \( \ell \to \infty \) only if \( T \to 0 \). Since \( \ell \) is finite at any finite temperature \( T \), the Ising model never orders (ordering happens if \( \ell \to 0 \), which in one dimensions only happens if \( T \to 0 \)).
2 One dimensional XY model

One dimensional XY model is defined via its partition function
\[ Z = \prod_{i=1}^{N} \int_{-\pi}^{\pi} d\phi_i \exp \left( \frac{J}{T} \sum_{i=1}^{N-1} \cos (\phi_i - \phi_{i-1}) \right). \] (2.2)

This model has a U(1) symmetry. Just like one dimensional Ising model, it does not order at any finite temperature.

Calculate the correlation function (as before, 1 \leq n < m \leq N).
\[ \langle e^{i\phi_n} e^{-i\phi_m} \rangle = \frac{1}{Z} \left[ \prod_{i=1}^{N} \int_{-\pi}^{\pi} d\phi_i \right] e^{i\phi_n} e^{-i\phi_m} \exp \left( \frac{J}{T} \sum_{i=1}^{N-1} \cos (\phi_i - \phi_{i-1}) \right). \] (2.3)

You may find the trick of changing variables to \( \xi_i = \phi_i - \phi_{i+1} \) for \( i = 1, 2, \ldots, N-1 \), useful. Recast this correlation function as \( \exp(-(m-n)/\ell) \) and find the correlation length \( \ell \).

3 Two dimensional lattice QED

Consider a U(1) lattice gauge theory placed in two dimensional square lattice. It consists of variables \( \sigma = e^{i\theta} \) on every link of the lattice, and the partition function
\[ Z = \prod_{i=1}^{N} \int_{-\pi}^{\pi} d\theta \exp \left( \frac{J}{T} \sum_{\text{plaquettes}} (1 - \cos(\theta + \theta - \theta - \theta)) \right). \] (3.3)

The product is over all the links of the two dimensional square lattice, \( \theta + \theta - \theta - \theta \) is a shorthand notation for the sums and differences of \( \theta \) over the plaquettes of the square lattice as adopted in lattice gauge theory. As one goes around the plaquette in the counter-clockwise direction, the links traversed in the positive direction (as defined by the direction of the axes of the cartesian reference frame) contribute \( +\theta \) and the links traversed in the negative direction contribute \( -\theta \) to the sum. This model has a U(1) gauge symmetry, defined by
\[ \theta \rightarrow \theta + \phi - \phi, \] (3.4)
where \( \phi \) are variables placed at the end points of the link where \( \theta \) is placed, with \( +\phi \) taken from the site at the positive end of the link, and \( -\phi \) from the site at the negative end of the link. The positive end is the one one encounters if one moves along the link in the
direction of the axes of the cartesian reference frame, and the negative end is the one in the opposite direction.

Wilson loop is defined as the product of the exponentials over the variables $\theta$ over a loop $C$,

$$W_C = \prod_C e^{\pm i\theta},$$

(3.5)

where again as one goes around the loop $C$ in the counter-clock-wise direction, the links traversed in the positive direction contribute $+i\theta$ and those in the negative direction contribute $-i\theta$ to the exponentials above. The expectation value of the Wilson loop is

$$\langle W_C \rangle = \frac{1}{Z} \left[ \prod_{-\pi}^{\pi} d\theta \right] W_C e^{\frac{i}{T} \sum_{\text{plaquettes}} (1-\cos(\theta+\theta-\theta))}.$$ 

(3.6)

It is always possible to choose a gauge where the variables $\theta$ placed on links in the, say, $y$-direction of the lattice, are equal to 0, and reduce this model to the integration over $\theta$ placed on the links pointing in the $x$-direction.

Consider a rectangular Wilson loop whose sides are of length $X$ and $Y$. Calculate the expectation value of the Wilson loop (it’s probably easiest to do working in the gauge above). You may need to rely on the solutions worked out in the previous problems of this problem set. What is the phase of the 2D lattice QED that you find as a result of this calculation?