1 Hamiltonian formalism for fields

For a particle the least action principle

\[ S = \int_{t_i}^{t_f} dt \, L(q, \dot{q}) \] (1.1)

quickly leads to the Euler-Lagrange equation

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}. \] (1.2)

In turn, these equations are equivalent to the Hamilton equations of motion. These are derived by first defining the momentum via

\[ p = \frac{\partial L(\dot{q}, q)}{\partial \dot{q}}. \] (1.3)

Then this equation is understood as the equation for \( \dot{q} \), and \( \dot{q} \) is found as a function of \( p \), and \( q \), that is, it becomes \( \dot{q}(p, q) \). Then the Hamiltonian is defined by

\[ H(p, q) = p\dot{q} - L. \] (1.4)

Finally, it is shown that the Euler-Lagrange equation can be rewritten as

\[ \dot{p} = -\frac{\partial H(p, q)}{\partial q}, \quad \dot{q} = \frac{\partial H(p, q)}{\partial p}. \] (1.5)

Indeed, first of all, we see that

\[ \frac{\partial H}{\partial p} = \frac{\partial (p\dot{q} - L)}{\partial p} = \dot{q} + \left( p - \frac{\partial L}{\partial q} \right) \frac{\partial \dot{q}}{\partial p} = \dot{q}, \] (1.6)

Second

\[ \frac{\partial H}{\partial q} = \left( p - \frac{\partial L}{\partial \dot{q}} \right) \frac{\partial \dot{q}}{\partial q} - \frac{\partial L}{\partial q} = -\frac{\partial L}{\partial q} = -\dot{p}, \] (1.7)

where the last equality follows from the Euler-Lagrange equation.
Now suppose we have a field theory where

$$S = \int dt d^d x \mathcal{L}(\phi, \partial \phi).$$  \hspace{1cm} (1.8)$$

Its Euler-Lagrange equations are

$$\frac{\delta S}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \sum_{\mu} \partial_{\mu} \left[ \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \right] = 0.$$  \hspace{1cm} (1.9)$$

Definition: \(X = \frac{\delta S}{\delta \phi}\) if

$$S[\phi + \delta \phi] - S[\phi] = \int dt d^d x X \delta \phi,$$  \hspace{1cm} (1.10)$$

for \(\delta \phi\) infinitesimally small.

An important claim is that we can define a momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}},$$  \hspace{1cm} (1.11)$$

solve this for \(\dot{\phi}\), define the Hamiltonian

$$H = \int d^d x \pi \dot{\phi} - \int d^d x \mathcal{L},$$  \hspace{1cm} (1.12)$$

so that the Euler-Lagrange equations become

$$\dot{\pi} = -\frac{\delta H}{\delta \phi}, \quad \ddot{\phi} = \frac{\delta H}{\delta \pi}.$$  \hspace{1cm} (1.13)$$

Show that the equations (1.13) follow from (1.9).

**Solution.** The solution is essentially redoing (1.6) and (1.7) with variational derivatives. We find, first of all,

$$\frac{\delta H}{\delta \pi(\vec{r})} = \frac{\delta}{\delta \pi(\vec{r})} \left( \int d^d x \left( \pi \dot{\phi} - \mathcal{L} \right) \right) = \dot{\phi}(\vec{r}) + \left( \pi - \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \frac{\partial \phi}{\partial \pi} = \frac{\delta}{\delta \pi(\vec{r})} \left( \int d^d x \left( \pi \dot{\phi} - \mathcal{L} \right) \right) = \dot{\phi}(\vec{r}).$$  \hspace{1cm} (1.14)$$

Then we find

$$\frac{\delta H}{\delta \phi(\vec{r})} = \frac{\delta}{\delta \phi(\vec{r})} \left( \int d^d x \left( \pi \dot{\phi} - \mathcal{L} \right) \right) = \left( \pi - \frac{\partial \mathcal{L}}{\partial \phi} \right) \frac{\partial \dot{\phi}}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial \phi} + \nabla \left[ \frac{\partial \mathcal{L}}{\partial \nabla \phi} \right] = -\dot{\pi}.$$  \hspace{1cm} (1.15)$$
Quantization of the complex scalar field

Solve problem 2.2 of the textbook (page 33).

**Solution.** The action is

\[
S = \int d^4x \left( \partial_{\mu} \phi^{*} \partial^{\mu} \phi - m^2 \phi^{*} \phi \right). \tag{2.1}
\]

Reminder: \( \partial_{\mu} \phi^{*} \partial^{\mu} \phi \) means \( \partial_t \phi^{*} \partial_t \phi - \partial_r \phi^{*} \partial_r \phi \).

(a) First of all, we find the momenta.

\[
\pi = \frac{\partial L}{\partial \dot{\phi}} = \partial_t \phi^{*}, \quad \pi^{*} = \frac{\partial L}{\partial \dot{\phi}^{*}} = \partial_t \phi. \tag{2.2}
\]

The Hamiltonian is

\[
H = \int d^3x \left( \pi \partial_t \phi + \pi^{*} \partial_t \phi^{*} - L \right) = \int d^3x \left( \pi^{*} \pi + \partial_r \phi^{*} \partial_r \phi + m^2 \phi^{*} \phi \right). \tag{2.3}
\]

The canonical commutation relations are

\[
[\pi(\vec{r}_1), \phi(\vec{r}_2)] = -i \delta(\vec{r}_1 - \vec{r}_2), \quad [\pi^{*}(\vec{r}_1), \phi^{*}(\vec{r}_2)] = -i \delta(\vec{r}_1 - \vec{r}_2). \tag{2.4}
\]

The rest of the fields commute. Upon quantization we replace \( \phi^{*} \rightarrow \phi^\dagger, \pi^{*} \rightarrow \pi^\dagger \) as appropriate for the operators.

(b) We first introduce

\[
\phi(\vec{p}) = \int d^3r \phi(\vec{r}) e^{-i\vec{p} \cdot \vec{r}}, \quad \phi^\dagger(\vec{p}) = \int d^3r \phi^{\dagger}(\vec{r}) e^{i\vec{p} \cdot \vec{r}}, \tag{2.5}
\]

\[
\pi(\vec{p}) = \int d^3r \pi(\vec{r}) e^{-i\vec{p} \cdot \vec{r}}, \quad \pi^\dagger(\vec{p}) = \int d^3r \pi^{\dagger}(\vec{r}) e^{i\vec{p} \cdot \vec{r}}. \tag{2.6}
\]

Just as in class, those satisfy

\[
[\pi(\vec{p}), \phi(\vec{q})] = -i \delta(\vec{p} + \vec{q})(2\pi)^3, \quad \left[ \pi^\dagger(\vec{p}), \phi^{\dagger}(\vec{q}) \right] = -i \delta(\vec{p} + \vec{q})(2\pi)^3. \tag{2.7}
\]

Then we write (since \( \phi \) is not real) with \( \omega_p = \sqrt{p^2 + m^2} \)

\[
a_\vec{p} = \frac{1}{2} \left( \sqrt{2 \omega_p} \phi(\vec{p}) + i \sqrt{2 \omega_p} \pi^\dagger(-\vec{p}) \right), \quad a_\vec{p}^\dagger = \frac{1}{2} \left( \sqrt{2 \omega_p} \phi^{\dagger}(\vec{p}) - i \sqrt{2 \omega_p} \pi(-\vec{p}) \right). \tag{2.8}
\]

It immediately follows that

\[
[a_\vec{p}, a_{\vec{q}}^\dagger] = \delta(\vec{p} - \vec{q})(2\pi)^d. \tag{2.9}
\]
Likewise

\[ b_\rho^\dagger = \frac{1}{2} \left( \sqrt{2\omega_\rho} \phi(-\vec{p}) - i \frac{\sqrt{2}}{\omega_\rho} \pi^\dagger(\vec{p}) \right), \quad b_\rho = \frac{1}{2} \left( \sqrt{2\omega_\rho} \phi^\dagger(-\vec{p}) + i \frac{\sqrt{2}}{\omega_\rho} \pi(\vec{p}) \right). \]  

This leads to

\[ [b_\rho, b_\eta^\dagger] = \delta(\vec{p} - \vec{q})(2\pi)^d. \]  

Moreover, \( a \) and \( a^\dagger \) commute with \( b \) and \( b^\dagger \).

Finally, we need to work out \( H \) in terms of these operators. We find

\[ H = \int \frac{d^3p}{(2\pi)^3} \left( a_\rho^\dagger a_\rho + b^\dagger_\rho b_\rho \right) \omega_\rho + \text{const}. \]  

(c) We find

\[ Q = \frac{i}{2} \int d^3x \left( \pi^\dagger \phi^\dagger - \pi \phi \right) = \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left( \pi^\dagger(\vec{p}) \phi^\dagger(-\vec{p}) - \pi(\vec{p}) \phi(-\vec{p}) \right) = \int \frac{d^3p}{(2\pi)^3} \left( a^\dagger_\rho a_\rho - b^\dagger_\rho b_\rho \right). \]  

Therefore, \( b \)-particles have an opposite charge to the charge of \( a \)-particles.

(d) The action for \( n \) complex scalar fields is

\[ S = \sum_{a=1}^n \int d^4x \left( \partial_\mu \phi^*_a \partial^\mu \phi_a - m^2 \phi^*_a \phi_a \right). \]  

This action is symmetric under the transformation

\[ \phi_a \rightarrow \sum_{b=1}^n U_{ab} \phi_b, \]  

where \( U \) is any (time and space independent) unitary matrix. That is, the matrix which satisfies

\[ U^\dagger U = 1. \]  

(2.15) is a shorthand for the following: Given an action in terms of \( \tilde{\phi} \),

\[ S = \sum_{a=1}^n \int d^4x \left( \partial_\mu \tilde{\phi}^*_a \partial^\mu \tilde{\phi}_a - m^2 \tilde{\phi}^*_a \tilde{\phi}_a \right), \]  

upon substitution of \( \tilde{\phi}_a = \sum_b U_{ab} \phi_b \), the new action is (2.14), that is, has exactly the same form as (2.17) if \( \phi \) is substituted for \( \tilde{\phi} \) in it.

A unitary matrix \( U \) is parametrized by \( n^2 \) independent parameters. For infinitesimal transformations one can parametrize

\[ U_{ab} = \delta_{ab} + i \sum_{i=1}^{n^2} \epsilon_i S_{ai} \]  

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where \( S^j \) are the generators of the \( U(N) \) group (and \( \epsilon_i \) are \( n^2 \) parameters). Therefore, an infinitesimal transformation of the fields is

\[
\delta_j \phi_a = i \epsilon_j \sum_{b=1}^{n} S^j_{ab} \phi_b,
\]

(2.19)

for \( j = 1, 2, \ldots, n^2 \) (no summation over \( j \) is implied, \( j \) is just one of those \( n^2 \) numbers). And the conserved quantities are given by (follows from Noether theorem, Eq (2.12) of the textbook)

\[
Q^j = \int d^3x \frac{i}{2} \left( \phi^*_a S^j_{ab} \pi^*_b - \pi_a S^j_{ab} \phi_b \right),
\]

(2.20)