Week 13: XY model at $d > 2$
1 XY model at low temperature

XY model is basically the O(2) model in the notation of the O(N) models. On every site of a lattice one has a two-dimensional vector of unit length and the interaction is proportional to the scalar product of two vectors on nearby sites. Parametrizing them by angles $\phi_r$, where $r$ labels the sites of the lattice, one has for the energy

$$\frac{E}{T} = -K \sum_{\langle r_1 r_2 \rangle} \cos(\phi_{r_1} - \phi_{r_2}).$$

(1.1)

The partition function is then given by

$$Z = \left[ \prod_j \int_{-\pi}^{\pi} d\phi_j \right] e^{K \sum_{\langle r_1 r_2 \rangle} \cos(\phi_{r_1} - \phi_{r_2})}.$$  

(1.2)

The notation $\langle r_1 r_2 \rangle$ as always implies that these are the neighboring points on the lattice, and we sum over all pairs of neighbors.

Above 2 dimensions, this model has a phase transition at a certain value of $K = \frac{J}{T}$. At a temperature below some critical temperature $T < T_c$, or equivalently at $K > K_c$, the spins order and mostly point in some chosen direction. This means that the correlation function goes to a constant at large distances. In terms of the angles $\phi$, the relevant correlation function is the cosine of the angle difference, because this truly measures whether the spins point in the same direction or not (here $r_1$ and $r_2$ are not the neighbors, just some two points on the lattice):

$$\langle \cos(\phi_{r_1} - \phi_{r_2}) \rangle \rightarrow \text{const}, \text{ when } |r_1 - r_2| \rightarrow \infty.$$  

(1.3)

This correlation function can be rewritten as (technically speaking, the real part of this is the correlation function we seek)

$$\left\langle e^{i\phi_{r_1} - i\phi_{r_2}} \right\rangle.$$  

(1.4)

It is not easy to calculate this correlation function. But suppose we at some extremely low temperature, or $K$ is very large. Then we imagine it is essentially impossible for the nearby spins to point in directions which are different by anything but a tiny difference. Then we can expand the cosine in the expression for the energy and write

$$Z = \left[ \prod_j \int_{-\pi}^{\pi} d\phi_j \right] e^{-\frac{K}{2} \sum_{\langle r_1 r_2 \rangle} (\phi_{r_1} - \phi_{r_2})^2}.$$  

(1.5)
Here a constant which came from the expansion of the cosine was, as always, dropped (since it multiplies the partition function by some irrelevant constant and does nothing else). As we often do, we may now employ continuous notations, replace differences by derivatives multiplied by a lattice spacing $a$, and the sum by the integral. The result reads

$$Z = \int \mathcal{D}\phi e^{-\frac{K}{2}a^{2-d}\int d^d r (\nabla \phi)^2}. \quad (1.6)$$

The correlation function we seek is given by

$$\langle e^{i\phi(r_1) - i\phi(r_2)} \rangle = \frac{1}{Z} \int \mathcal{D}\phi e^{i\phi(r_1) - i\phi(r_2) - \frac{K}{2}a^{2-d}\int d^d r (\nabla \phi)^2}. \quad (1.7)$$

Such a correlation function is expressed in terms of a Gaussian integral. But we know how to compute a Gaussian integral. In the notes on Gaussian integral, we derived the following formula (the formula was derived in the discrete notations, this is its analogue in the continuous notations, and the formula is in Week 8 notes, (2.17) to (2.23)) :

$$\int \mathcal{D}\phi e^{\int d^d r J(r) \phi(r) - \frac{1}{2} \int d^d r d^d r' \phi(r) \hat{O}(r-r') \phi(r)} = e^{\frac{1}{2} \int d^d r d^d r' J(r) \hat{O}^{-1}(r-r')J(r')} \int \mathcal{D}\phi e^{-\frac{1}{2} \int d^d r d^d r' \phi(r) \hat{O}(r-r') \phi(r)}. \quad (1.8)$$

Here $\hat{O}$ is any operator (or a matrix in the discrete notations). In our case, $\hat{O} = -K a^{2-d} \Delta$. ($\Delta$ is a Laplacian). This requires some clarification. Matrices, as well as operators, have two indices. Just as the result of multiplying a matrix $M_{ij}$ by a vector $x_j$ is

$$y_i = \sum_j M_{ij} x_j, \quad (1.9)$$

in the same way the result of applying an operator to a function is

$$y(r) = \int d^d r' \hat{O}(r-r') x(r'). \quad (1.10)$$

However, often we deal with operators such as derivatives or Laplacians, which don’t seem to depend on two coordinates. This is because the “canonical way” to represent an operator such as a Laplacian would be

$$\Delta \equiv \delta(r-r') \Delta_{r'}. \quad (1.11)$$

Here $\Delta_{r'}$ is a Laplacian which acts on the coordinates $r'$ and $\delta$ is a Dirac delta-function. Indeed, then

$$y(r) = \Delta x(r) = \int d^d r' \delta(r-r') \Delta_{r'} x(r'). \quad (1.12)$$
In other words, it is not important to keep track of the operator’s dependence on \( r \) and \( r' \). Delta-function is short ranged, and applying such operator to a function, such as \( x(r) \), is just like applying a Laplacian to this function.

However, while \( \hat{O} \) is a short range operator, that is, is proportional to a delta-function \( \delta(r - r') \), its inverse \( \hat{O}^{-1} \) is by no means short ranged. So it is important to keep track of its dependence on \( r \) and \( r' \).

At the same time, \( J = i\delta(r - r_1) - i\delta(r - r_2) \). Here \( \delta \) is the Dirac delta-function.

To apply this Gaussian formula, we need to know what the inverse of a Laplacian is. Again, let’s study this on an example of a matrix, \( M_{jk} \). Suppose we would like to calculate \( M^{-1}_{jk} \). This can be done if we know the eigenvalues and the eigenvectors of this matrix. Indeed, suppose

\[
\sum_k M_{jk} \psi_k^{(n)}(r) = \lambda^{(n)} \psi_j(r).
\]

(1.13)

Here \( \psi_k^{(n)} \) is an \( n \)-th eigenvector (normalized, so that \( \sum_k (\psi_k^{(n)})^* \psi_k^{(n)} = 1 \)), with the eigenvalue \( \lambda^{(n)} \). Then the inverse of the matrix is given by

\[
M^{-1}_{jk} = \sum_n \frac{\psi_j^{(n)}(r) (\psi_k^{(n)})^*}{\lambda_n}.
\]

(1.14)

This is a famous formula very useful in the kind of calculations about to be done here. It can be checked that \( M^{-1} \), as defined above, indeed satisfies

\[
\sum_k M^{-1}_{jk} M_{kl} = \delta_{jl}.
\]

(1.15)

Now the eigenvalues of a Laplacian are \(-p^2\), with eigenfunctions being \( e^{ipr}/\sqrt{V} \). Here \( V \) is the volume of the system. It appears because these eigenfunctions need to be normalized, or

\[
\frac{1}{V} \int d^d r e^{ipr} e^{-ipr} = \frac{1}{V} \int d^d r = 1.
\]

(1.16)

So the inverse of the Laplacian is actually

\[
\hat{O}^{-1}(r - r') = \frac{V}{VK a^{2-d}} \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(r-r')}}{p^2} \sim \frac{1}{Ka^{2-d} |r - r'|^{d-2}}.
\]

(1.17)

Here we use that the summation over wave vectors \( p \) is known to be equivalent to integration over \( V d^d p/(2\pi)^d \). Notice that \( V \) from this and from the normalization of the eigenfunctions conveniently cancel, never to appear again.
We discussed that this integral is proportional to $r^{2-d}$ many times previously; the proportional means up to some numerical constant which is not relevant. This means that the correlation function we wanted to compute becomes

\[
\langle e^{i\phi_{r_1} - i\phi_{r_2}} \rangle = \frac{1}{Z} \int D\phi \ e^{i\phi(r_1) - i\phi(r_2)} - \frac{K}{2} a^{2-d} \int d^dr (\nabla \phi)^2
\]

\[
= e^{-\frac{1}{2} \int d^4r d^4r' [\delta(r-r_1) - \delta(r-r_2)] |\hat{O}^{-1}(r-r')| \delta(r'-r_1) - \delta(r'-r_2)]} = e^{-\hat{O}^{-1}(0) + \hat{O}^{-1}(r_1 - r_2)}. \tag{1.18}
\]

It was important that $\hat{O}(r_1 - r_2) = \hat{O}(r_2 - r_1)$ in the derivation of the last line.

Naively it appears that $\hat{O}^{-1}(0)$ is infinity. However, we should remember that the integral in (1.17) is actually over momenta which cannot exceed $p \sim \pi/a$, where $a$ is a lattice spacing. So $\hat{O}(r - r')$ can be calculated in case when $r = r'$ by substituting $|r - r'| = a$, with the result

\[
\hat{O}^{-1}(0) = \frac{1}{K}. \tag{1.19}
\]

This finally gives

\[
\langle e^{i\phi_{r_1} - i\phi_{r_2}} \rangle = e^{-\frac{1}{K} + \frac{a^{d-2}}{K^{d-2}}} \tag{1.20}
\]

At large distances this goes to a constant, as long as $d > 2$. This completes the proof that this model is ordered at low temperatures.

At higher temperature there should be a phase transition separating the ordered phase with a disordered phase. The transition becomes mean field if $d \geq 4$, as always. The techniques used here, however, do not apply at temperatures so high when the transition happens. These techniques, which predict an ordered phase no matter what $K$ is, are restricted to low temperature.