Advanced Statistical Mechanics

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Week 3: 2D and 1D transverse field Ising models, exact solution
1 2D Ising Model: Peierls Argument

2D Ising model is defined on a 2D lattice. Each spin of the 2D Ising model lives on a 2-dimensional space, thus it has two coordinates. Let’s call them horizontal and vertical coordinates, and let’s denote them $x$ and $\tau$. So each spin can be written as $\sigma^x_\tau$. Suppose the Ising model has length $N_h$ in the horizontal direction and $N_v$ in the vertical direction, so all together it has $N_h N_v$ spins.

The energy of the Ising model can be written as

$$E = - \sum_{x=1,...,N_h; \tau=1,...,N_v} \left[ J_h \sigma^x_\tau \sigma^x_{x+1} + J_v \sigma^\tau_x \sigma^\tau_{x+1} \right]$$

(1.1)

and the partition function is

$$Z = \sum_{\sigma_i = \pm 1} e^{-\frac{E}{T}}.$$  

(1.2)

For generality we introduce two distinct couplings, $J_h$ and $J_v$ in the horizontal and vertical directions, although in the classic Ising model $J_h = J_v = J$. We also need to specify what happens at the boundary. Let’s take the periodic boundary conditions. To reduce clutter, let us adopt this common notations:

$$\sigma^x_{N_v+1} \equiv \sigma^x_1, \sigma^\tau_{N_h+1} \equiv \sigma^\tau_1.$$  

(1.3)

In this way, the periodic boundary conditions are automatically encoded in (1.1).

Peierls came up with a hand-waving argument which supports that at temperature below some critical temperature $T_c$, the Ising spins order. Let us go over this argument quickly in case where $J_h = J_v = J$, and $J/T = K$.

A droplet of negative spins in the sea of positive spins has energy

$$E_0 + 2JL,$$  

(1.4)

where $E_0$ is the energy of all spins pointing up, or $E_0 = 2J N_v N_h$ ($2N_v N_h$ is the total number of bonds on the square lattice of our size). Here $L$ is the length of the boundary between a droplet of spins pointing down, and the rest of the spins pointing up (each bond where spin at one end points up and a spin at the other point points down has energy $+J$, which differs from its energy when both spins point up (which is $-J$) by $2J$. So $2J$ times the number of bonds with opposite spins at ends is the energy of the droplet, relative to $E_0$. There are $L$ such bonds, so the total is $E_0 + 2JL$).

Its contribution to $Z$ is then $e^{-\frac{E_0}{T} - 2KL}$. On the other hand, there are $c^L$ ways to position a droplet. This produces a contribution to $Z$ of the order of $e^{-E_0/T} e^{L \ln c - 2KL}$. 

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which, at large $K$ (small temperature, as $K = J/T$), will always be small. In other words, 
the free energy of the droplet is 

$$F = E - TS = 2JL - LT \ln c - E_0$$  \hspace{1cm} (1.5)$$

and its minimum at $T < 2J/\ln c$ is achieved at zero $L$. This means at $T < 2J/\ln c$, there 
is going to be no droplets around and the spins will all be positive.

At $T > 2J/\ln c$ the minimum of free energy is at very large $L$, in other words, these 
droplets will be around, they will be large, and eventually they will blur the difference 
between spin-ups and spin-downs, so the spins will not be ordered.

The transition happens at critical temperature $T_c$ which this argument says is $2J/\ln c$. 
This is just a rough estimate, in practice this temperature is proportional to $J$ with some 
numerical coefficient in front.

If $J_h \neq J_v$, there is still a transition temperature $T_c$ which can be expressed in terms 
of these. This expression is known exactly, but its form is not illuminating and so won’t 
be given here.

2 Transfer Matrix and the Hamiltonian

We rewrite the partition function as a transfer matrix. This is a little more involved than 
in 1D.

In one dimension, the transfer matrix has two indices, $\sigma_1$ and $\sigma_2$. In 2D, we designate 
all spins along one row of spins as one index of the transfer matrix, and all spins along an 
adjacent row as the other index. Thus a transfer matrix is a matrix of the size $2^N_h$ by $2^N_h$.

The transfer matrix takes the form

$$T_{\sigma_1^1, \sigma_1^2, ..., \sigma_N^h} = e^{\sum_{x=1}^{N_h} \left( K_v \sigma_x^1 \sigma_{x+1}^1 + K_v \sigma_x^2 \sigma_{x+1}^2 \right)}.$$  \hspace{1cm} (2.1)$$

Here we introduced the natural notations

$$K_h = \frac{J_h}{T}, \quad K_v = \frac{J_v}{T}.$$  \hspace{1cm} (2.2)$$

Again, it has two indices, labeling its rows and columns. Each index goes from 1 to $2^N_h$. 
However, we denote the index not by one integer going between these two values, but 
rather by a string of integers $\sigma_1^1, \sigma_2^1, \sigma_3^1, ..., \sigma_N^1$ (lower index) and $\sigma_1^2, \sigma_2^2, \sigma_3^2, ..., \sigma_N^2$ (upper 
index), each $\sigma$ taking values $\pm 1$, all for $2^{N_v}$ distinct strings of integers.
Then we can write the partition function in the following way

\[
\sum_{\sigma=\pm 1} e^{\sum_{x=1}^{N_v} \sum_{\tau=1}^{N_h} [K_h \sigma_1^x \sigma_1^{x+1} + K_v \sigma_2^{x} \sigma_2^{x+1}]} = \sum_{\sigma=\pm 1} T_{\sigma_1^1 \sigma_1^{N_h}} T_{\sigma_1^1 \sigma_2^1} \cdots T_{\sigma_1^1 \sigma_2^{N_v}} (2.3)
\]

This in turn can be interpreted as

\[
Z = \text{Tr} \ T^{N_v} = \sum_n \lambda_n^{N_v} \approx \lambda_0^{N_v}, \quad (2.4)
\]

where \(\lambda_n\) are the eigenvalues of \(T\) (there are going to be \(2^{N_h}\) of those because this matrix is \(2^{N_h}\) by \(2^{N_h}\)) and \(\lambda_0\) is the largest of these eigenvalues.

So the task is to diagonalize the matrix \(T\), written explicitly in (2.1). This is a complicated matrix and it’s not easy to diagonalize. However, there is a particular limit in which the matrix considerably simplifies. This is the limit \(K_v \gg 1, K_h \ll 1\). Let us derive the transfer matrix in this limit.

To do that, we first of all split the transfer matrix into a product of two matrices, the one responsible for the horizontal part of the transfer matrix, and the one responsible for the vertical part, in the following way

\[
T_{\sigma_1^1 \sigma_2^1 \cdots \sigma_1^{N_h}} = \sum_{\bar{\sigma}} [T_h]_{\sigma_1^1 \sigma_2^1 \cdots \sigma_1^{N_h}} [T_v]_{\bar{\sigma}_1^1 \bar{\sigma}_2^1 \cdots \bar{\sigma}_1^{N_h}}. \quad (2.5)
\]

Here

\[
[T_h]_{\sigma_1^1 \sigma_2^1 \cdots \sigma_1^{N_h}} = e^{K_h \sum_{x=1}^{N_h} \sigma_1^x \sigma_1^{x+1} \delta_1^{x+1} \sigma_2^x \delta_2^x \cdots \delta_1^1 \sigma_2^{N_v}} \quad (2.6)
\]

and

\[
[T_v]_{\bar{\sigma}_1^1 \bar{\sigma}_2^1 \cdots \bar{\sigma}_1^{N_h}} = e^{K_v \sum_{x=1}^{N_v} \sigma_1^x \sigma_2^x}. \quad (2.7)
\]

It should be clear that multiplying \(T_h\) by \(T_v\), as is done in (2.5), gives \(T\), as in (2.1).

Next we employ the following strategy. We will show that, under conditions \(K_v \gg 1\) and \(K_h \ll 1\), each of these two matrices can be written as

\[
T_h = e^{-H_h}, \quad T_v = e^{-H_v} \quad (2.8)
\]

where \(H_h\) and \(H_v\) are very small matrices. Now in general for two matrices \(A\) and \(B\), \(e^A e^B \neq e^{A+B}\). But for two very small matrices

\[
e^{-H_h} e^{-H_v} \approx (1 - H_h) (1 - H_v) \approx 1 - H_h - H_v \approx e^{-H_h - H_v}. \quad (2.9)
\]

So we will be able to derive \(T\) in this way.
Let’s follow this route. Take $T_h$ first. Suppose $K_v \ll 1$. Then we can Taylor-expand the exponential:

$$[T_h]_{\sigma_1^1,\sigma_2^1,\ldots,\sigma_N^1} = \delta_{\sigma_1^1} \delta_{\sigma_2^1} \ldots \delta_{\sigma_N^1} + K_h \sum_{x=1}^{N_h} \left[ \delta_{\sigma_1^1} \delta_{\sigma_2^1} \ldots \delta_{\sigma_N^1} \right]$$

We rewrite this with the help of the Pauli matrices

$$[T_h]_{\sigma_1^1,\sigma_2^1,\ldots,\sigma_N^1} = I + K_h \sum_{x=1}^{N_h} \left[ \sigma_1^1 \sigma_2^1 \ldots \sigma_N^1 \right]$$

Now the first term in this sum is nothing but an identity matrix (a very large, $2^{N_h}$ by $2^{N_h}$ identity matrix). The second term represents a sum of terms, each is a product of many little identities acting on all spins except $\sigma_1^1$ and $\sigma_2^1$, which are acted upon by Pauli matrices instead. A shorthand notation is often introduced, where all these little Kronecker deltas are omitted (just as multiplication by 1 is usually omitted), which leads to the following simplified notation (yet completely equivalent (2.12))

$$[T_h]_{\sigma_1^1,\sigma_2^1,\ldots,\sigma_N^1} = I + K_h \sum_{x=1}^{N} r_x^3 x x + 1.$$  

Here $I$ stands for this very large identity matrix.

The final step is to recall that $K_h \ll 1$, so we can write

$$T_h = e^{K_h \sum_{x=1}^{N} r_x^3 x x + 1}.$$  

So we accomplished the task of writing $T_h$ in the form $e^{-H_h}$, where

$$H_h = -K_h \sum_{x=1}^{N} r_x^3 x x + 1.$$  

Now on to $T_v$. $T_v$ will be transformed in the following way. We first write

$$[T_v]_{\sigma_1^2,\sigma_2^2,\ldots,\sigma_N^2} = \prod_{x=1}^{N_h} e^{K_v \sigma_1^2}.$$

Let us study each matrix in this product.

$$e^{K_v \sigma_1^2}$$
is a two by two matrix, which explicitly takes the familiar (from 1D Ising model) form

\[
\begin{pmatrix}
e^{K_v} & e^{-K_v} \\
e^{-K_v} & e^{K_v}
\end{pmatrix}
\sigma_x^2 \sigma_x^1 =
\begin{pmatrix}
e^{K_v} & e^{-K_v} \\
e^{-K_v} & e^{K_v}
\end{pmatrix}
\sigma_x^2 \sigma_x^1.
\] (2.18)

This matrix is close to identity if \( K_v \gg 1 \). In this case we can write it as

\[
\begin{pmatrix}
e^{K_v} & e^{-K_v} \\
e^{-K_v} & e^{K_v}
\end{pmatrix} = e^{K_v} \begin{pmatrix}
1 & e^{-2K_v} \\
e^{-2K_v} & 1
\end{pmatrix} = e^{K_v} \left( \delta + e^{-2K_v} \tau^1 \right) \approx e^{K_v} e^{e^{-2K_v} \tau^1}.
\] (2.19)

Here the natural notations are

\[
\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (2.20)

Therefore, we can write

\[
\begin{pmatrix}
e^{K_v} & e^{-K_v} \\
e^{-K_v} & e^{K_v}
\end{pmatrix} \sigma_x^2 \sigma_x^1 = e^{K_v} e^{e^{-2K_v} \tau^1}.
\] (2.21)

Here the index \( x \) on \( \tau^1_x \) serves as a reminder that \( \tau^3 \) is applied to the \( x \)-th spin.

Now we can write

\[
\begin{pmatrix}
e^{K_v} & e^{-K_v} \\
e^{-K_v} & e^{K_v}
\end{pmatrix} \sigma_x^2 \sigma_x^1 \sigma_x^2 = e^{N_h K_v} \prod_{x=1}^{N_h} e^{e^{-2K_v} \tau^1_x}.
\] (2.22)

Finally, we recall that \( K_v \gg 1 \), so \( e^{-2K_v} \ll 1 \), so all these exponentials can be combined into a big sum.

\[
\begin{pmatrix}
e^{K_v} & e^{-K_v} \\
e^{-K_v} & e^{K_v}
\end{pmatrix} \sigma_x^2 \sigma_x^1 \sigma_x^2 = e^{N_h K_v} e^{e^{-2K_v} \sum_{x=1}^{N_h} \tau^1_x}.
\] (2.23)

So we succeeded in writing

\[
T_v = e^{K_v N_h} e^{-H_v}, \quad H_v = -e^{-2K_v} \sum_{x=1}^{N_h} \tau^1_x.
\] (2.24)

This completes the derivation of \( T_v \). Finally, we put everything together. We introduce convenient notations

\[
\gamma = e^{-2K_v}, \quad \beta = K_h.
\] (2.25)

This gives

\[
T = e^{K_v N_h} e^{-H},
\] (2.26)

where

\[
H = -\gamma \sum_{x=1}^{N_h} \tau^1_x - \beta \sum_{x=1}^{N_h} \tau^3_x \tau^3_{x+1}.
\] (2.27)
This Hamiltonian is called the 1D transverse field Ising model.

Indeed, this is a quantum mechanical Hamiltonian, which acts on quantum spins aligned in a 1D chain. \( \tau_x \) are spin operators (Pauli matrices) acting on \( x \)-th specific spin. We have a term \( \tau^3_x \tau^3_{x+1} \) which describes two spins trying to align in the same direction (\( z \)-direction) and a term \( \gamma \tau^1_x \) which corresponds to a magnetic field acting in the \( x \) direction. So it’s a 1D quantum spin chain with a magnetic field in the transverse direction.

Our goal is now to study this \( H \) and find its ground and excited states. This will solve the 2D Ising model, at least in the regime where \( K_v \gg 1, K_h \ll 1 \).

### 3 Ordered and Disordered Regimes

#### 3.1 Ordered regime

If \( \gamma \ll \beta \), then the Hamiltonian is approximately (neglect the \( \gamma \) term)

\[
H = -\beta \sum_{x=1}^{N_h} \tau^3_x \tau^3_{x+1}.
\]  
(3.1)

Then it has two ground states,

\[
|0\rangle = \prod_{x=1}^{N_h} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_x, \quad \text{or} \quad |0\rangle = \prod_{x=1}^{N_h} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i.
\]  
(3.2)

The notations here correspond to a wave function (a "spinor") for spins on sites \( x \). These are spins all pointing up or down. In the spirit of spontaneous symmetry breaking the spins choose one of these two states and ignore the other. Suppose the ground state corresponds to all spins up. The energy is

\[
E_0 = -\beta N_h.
\]  
(3.3)

An excited states would correspond to taking one spin and making it point down. This would correspond to the energy

\[
E_1 = -\beta N_h + 2\beta.
\]  
(3.4)

Of course, there are many such states, since any spin out of \( N_h \) spins, can be made to point down. But this does not matter.

We see that \( E_1 - E_0 = 2\beta \). Now since \( T = e^{-H} \), this leads to the eigenvalues of the transfer matrix

\[
\lambda_0 = e^{-E_0}, \quad \lambda_1 = e^{-E_1},
\]  
(3.5)
and the correlation length is
\[ \xi = \frac{1}{\ln \frac{N}{M}} = \frac{1}{E_1 - E_0} = \frac{1}{2\beta}. \] (3.6)

It is finite.

### 3.2 Disordered regime

Suppose now \( \gamma \gg \beta \). Neglect \( \beta \) to obtain
\[ H = -\gamma \sum_{x=1}^{N_h} \tau^1_x. \] (3.7)

The ground state takes the form
\[ |0\rangle = \prod_{x=1}^{N_h} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_x. \] (3.8)

This is because \((1, 1)\) is the eigenvector for \( \tau^1 \) with the eigenvalue 1. This corresponds to the energy
\[ E = -\gamma N_h. \] (3.9)

An excited state would correspond to taking one spin and replacing its spinor by \((1, -1)\) (again, there are many such excited states). This is an eigenvector of \( \tau^1 \) but with an eigenvalue \(-1\). So this gives
\[ E_1 = -\gamma N_h + 2\gamma. \] (3.10)

So we see that
\[ \xi = \frac{1}{E_1 - E_0} = \frac{1}{2\gamma}. \] (3.11)

In the next section, we are going to solve our problem exactly. We will see that the exact expression is
\[ \xi = \frac{1}{2|\gamma - \beta|}. \] (3.12)

This matches (3.6) and (3.11) in the limits \( \gamma \ll \beta \) and \( \beta \ll \gamma \) respectively.

### 4 Jordan-Wigner transformation

Now we will solve the Hamiltonian (2.27) exactly. First of all we do a unitary transformation so that
\[ H = -\gamma \sum_{x=1}^{N_h} \tau^2_x - \beta \sum_{x=1}^{N_h} \tau^1_x \tau^1_{x+1}. \] (4.1)
Corresponding transformation is given by \( U = \frac{1}{\sqrt{2}} (1 + i\tau^2) \), and

\[
U^\dagger \tau^3 U = -\tau^1, \quad U^\dagger \tau^1 U = \tau^3.
\] (4.2)

Then, it is convenient to introduce

\[
b_x = \frac{1}{2} \left( \tau_x^1 + i\tau_x^2 \right), \quad b_x^\dagger = \frac{1}{2} \left( \tau_x^1 - i\tau_x^2 \right)
\] (4.3)

They satisfy

\[
b_x^2 = b_x^\dagger b_x = 0, \quad b_x^\dagger b_x + b_x b_x^\dagger = 1 \quad (4.4)
\]

In this way they look like creation and annihilation operators. The transformations expressing \( \tau \) in terms of \( b \) read

\[
\tau_x^1 = b_x^\dagger + b_x, \quad \tau_x^2 = i(b_x^\dagger - b_x), \quad \tau_x^3 = -i/2 \left[ \tau_x^1, \tau_x^2 \right] = b_x b_x^\dagger - b_x^\dagger b_x.
\] (4.5)

Now we introduce the Jordan-Wigner string

\[
a_x = e^{i\pi \sum_{1\leq y<x} b_y^\dagger b_y} b_x, \quad a_x^\dagger = e^{-i\pi \sum_{y<x} b_y^\dagger b_y} b_x^\dagger
\] (4.6)

Then not only \( a \) obey creation and annihilation relations on a given site, but also

\[
a_x^\dagger a_y + a_y a_x^\dagger = 0, \quad x \neq y,
\] (4.7)

and so do \( a_x, a_y \) and \( a_x^\dagger, a_y^\dagger \).

Therefore, these are now fermions, called Jordan-Wigner fermions. Therefore, the Hamiltonian becomes (taking into account the Jordan-Wigner string which leads to the minus sign in the term \( a_x^\dagger - a_x \))

\[
H = \gamma \sum_{x=1}^{N_h} \left( a_x^\dagger a_x - a_x a_x^\dagger \right) - \beta \sum_{x=1}^{N_h} \left( a_x^\dagger - a_x \right) \left( a_{x+1}^\dagger + a_{x+1} \right)
\] (4.8)

This is the so-called Bogoliubov Hamiltonian, studied in superconductivity. This Hamiltonian can now be diagonalized using standard methods.

5 Diagonalizing the Hamiltonian - the straightforward method

To diagonalize this Hamiltonian we go to the plane wave basis

\[
a_x = \frac{1}{\sqrt{N_h}} \sum_k e^{ikx} a_k, \quad a_x^\dagger = \frac{1}{\sqrt{N_h}} \sum_k e^{-ikx} a_k^\dagger
\] (5.1)
Substituting this into (4.8) gives
\[ H = \gamma \sum_k \left( a_k^\dagger a_k - a_k a_k^\dagger \right) - \beta \sum_k \left( a_k^\dagger - a_{-k} \right) \left( a_{-k}^\dagger + a_k \right) e^{ik}. \] (5.2)

Here the sum over \( k \) goes from \(-\pi\) to \( \pi\), as usual (this is called the Brillouin zone), in steps of \( 2\pi/N \). This Hamiltonian clearly splits into terms coupling the modes \( k \) and \(-k\) for each positive \( k \). Therefore, we can write it as
\[ H = \sum_{k>0} H_k, \] (5.3)

where
\[ H_k = 2 \left( \gamma - \beta \cos k \right) \left( a_k^\dagger a_k - a_{-k} a_{-k}^\dagger \right) - 2i\beta \sin k a_k^\dagger a_{-k}^\dagger + 2i\beta \sin k a_{-k} a_k^\dagger. \] (5.4)

\( H_k \) can be written in a convenient matrix form
\[ H_k = 2 \begin{pmatrix} \gamma - \beta \cos k & -i\beta \sin k \\ i\beta \sin k & -\beta + \beta \cos k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}. \] (5.5)

This matrix has the “Bogoliubov-de-Gennes” form from the theory of superconductivity. It is a little inconvenient that it has imaginary entries, but these imaginary entries can be completely eliminated if one does a transformation
\[ a_{-k} = i\tilde{a}_{-k}, \quad a_{-k} = -i\tilde{a}_{-k}. \] (5.6)

The new variable \( \tilde{a}_{-k} \) are as good a fermionic variable as the old one, and the Hamiltonian becomes
\[ H_k = 2 \left( a_k^\dagger \quad \tilde{a}_{-k} \right) \begin{pmatrix} \gamma - \beta \cos k & \beta \sin k \\ \beta \sin k & -\beta + \beta \cos k \end{pmatrix} \begin{pmatrix} a_k \\ \tilde{a}_{-k}^\dagger \end{pmatrix}. \] (5.7)

Now the Hamiltonian can be diagonalized by a “Bogoliubov transformation”. This transformation reads
\[ a_k = \cos(\phi_k) c_k + \sin(\phi_k) c_{-k}^\dagger, \quad \tilde{a}_{-k} = -\sin(\phi_k) c_k + \cos(\phi_k) c_{-k}^\dagger. \] (5.8)

Here \( k > 0 \). It is clear from this that \( \phi_k = -\phi_{-k} \). Importantly, these transformations preserve the fermionic anticommutation relations. For example,
\[ \{ a_k, a_k^\dagger \} = \{ \cos(\phi_k) c_k + \sin(\phi_k) c_{-k}^\dagger, \sin(\phi_k) c_{-k} + \cos(\phi_k) c_k^\dagger \} = \cos^2 \phi_k + \sin^2 \phi_k = 1, \] (5.9)
\{ a_k, \tilde{a}_{-k} \} = \{ \cos(\phi_k) c_k + \sin(\phi_k) c^\dagger_{-k}, \cos(\phi_k) c_{-k} - \sin(\phi_k) c^\dagger_k \} = -\cos \phi_k \sin \phi_k + \cos \phi_k \sin \phi_k = 0, \quad (5.10)

and so on (to compute the left hand side, we assume that \( c_k \) satisfy the fermionic anticommutation relations, in other words, if \( c_k \) are fermions, then so are \( a_k \)). This Bogoliubov transformation is nothing but the orthogonal rotation of the matrix, in other words,

\[
\begin{pmatrix}
  a_k \\
  \tilde{a}_{-k}
\end{pmatrix}
= \begin{pmatrix}
  \cos \phi_k & \sin \phi_k \\
  -\sin \phi_k & \cos \phi_k
\end{pmatrix}
\begin{pmatrix}
  c_k \\
  c^\dagger_{-k}
\end{pmatrix}.
\]

\quad (5.11)

The Hamiltonian, when expressed in terms of \( c_k \), becomes

\[
H_k = 2 \left( c^\dagger_k c_{-k} \right) U \left( \begin{array}{cc}
\gamma - \beta \cos k & \beta \sin k \\
\beta \sin k & -\gamma + \beta \cos k
\end{array} \right) U \left( \begin{array}{c}
c_k \\
c^\dagger_{-k}
\end{array} \right), \quad U = \begin{pmatrix}
\cos \phi_k & \sin \phi_k \\
-\sin \phi_k & \cos \phi_k
\end{pmatrix}.
\]

\quad (5.12)

We can now choose the orthogonal matrix \( U \) in such a way that the Hamiltonian becomes diagonal. Its eigenvalues are \( h_k \) and \(-h_k\) where

\[
h_k = 2 \sqrt{(\gamma - \beta \cos k)^2 + \beta^2 \sin^2 k} = 2 \sqrt{\gamma^2 + \beta^2 - 2\gamma \beta \cos k} = 2 \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \sin^2 \frac{k}{2}}.
\]

\quad (5.13)

This brings the Hamiltonian to the form

\[
H_k = \begin{pmatrix}
  c^\dagger_k & c_{-k}
\end{pmatrix} \begin{pmatrix}
  h_k & 0 \\
  0 & -h_k
\end{pmatrix} \begin{pmatrix}
  c_k \\
  c^\dagger_{-k}
\end{pmatrix}.
\]

\quad (5.14)

Finally, this means

\[
H = \sum_{k>0} 2 \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \sin^2 \frac{k}{2}} \left( c^\dagger_k c_k - c_{-k} c^\dagger_{-k} \right).
\]

\quad (5.15)

This is a very interesting Hamiltonian. In its ground state all fermionic states are empty. The ground state energy is then

\[
E_0 = -2 \sum_{k>0} \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \sin^2 \frac{k}{2}} = -N_h \int_0^\pi \frac{dk}{2\pi} \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \sin^2 \frac{k}{2}}.
\]

\quad (5.16)

To create an excitation, we populate one fermionic state, either with positive momentum \( q \) or with negative momentum \(-q\). That increases the energy by

\[
E_q - E_0 = 2 \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \sin^2 \frac{q}{2}}.
\]

\quad (5.17)

The minimum energy increase occurs if we populate the state with \( q = 0 \). This gives

\[
E_{q=0} - E_0 = 2 |\gamma - \beta|,
\]

\quad (5.18)
and leads to the previously advertised result for the correlation length.

\[ \xi = \frac{1}{E_{q=0} - E_0} = \frac{1}{2 |\gamma - \beta|}. \]  

The point where \( \gamma = \beta \) is the transition point. At this point the correlation length is infinity.

Interestingly, at small \( q \) the “single fermion” excitation energy has a relativistic form

\[ (E_q - E_0)^2/4 = (\gamma - \beta)^2 + \gamma \beta q^2. \]  

Here \( |\gamma - \beta| \) plays the role of a relativistic “mass”. That’s why the critical point \( \gamma = \beta \) is often called “massless”, while the phases with gaps in the spectrum are often called “massive”.

Finally, we observe that this Hamiltonian (5.15) represents the Majorana fermions, not the usual “Dirac” fermions. That’s because it represents the positive momentum particles whose energy increases with \( k \), so they move to the right. But also it has negative momentum particles whose energy increases as \( k \) becomes more negative, so they move to the left.

Real-world (“Dirac”) fermions would’ve been moving either left and/or right at all \( k \), not just at positive \( k \). These kinds of special fermions which have their momentum restricted to either positive values (\( c_k \) moving to the right) or negative values (\( c_{-k} \) moving to the left) are called Majorana fermions.

### 6 Diagonalizing the Hamiltonian - alternative scheme

Here is another way to diagonalize the Hamiltonian, faster but involving a more convoluted mathematical construction. To diagonalize it, we employ the Bogoliubov transformations,

\[ \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) = U \left( \begin{array}{c} a \\ a^\dagger \end{array} \right) = \left( \begin{array}{cc} A & B \\ B^* & A^* \end{array} \right) \left( \begin{array}{c} a \\ a^\dagger \end{array} \right), \quad \left( \begin{array}{c} c^\dagger \\ c \end{array} \right) = \left( \begin{array}{cc} a^\dagger & a \end{array} \right) U^\dagger. \]  

(6.1)

the anticommutation relations. This is shorthand notations, in components this looks like this: \( U_{ij} \) is a matrix, and

\[ c_x = \sum_{y=1}^{N_h} \left[ A_{xy} a_y + B_{xy} a_y^\dagger \right], \quad c_x^\dagger = \sum_{y=1}^{N_h} \left[ B_{xy}^* a_y + A_{xy}^* a_y^\dagger \right]. \]  

(6.2)

Let us prove that \( U \) must be a unitary matrix. This is a consequence of the anticommutation relations. Let us introduce

\[ \psi_j = \left( \begin{array}{c} a \\ a^\dagger \end{array} \right). \]  

(6.3)
This is again a shorthand notation. Here \( j \) goes from 1 to \( 2N_h \). If \( 1 \leq j \leq N_h \), then \( \psi_j = a_j \). If \( N_h < j \leq 2N_h \), then \( \psi_j = a_{j-N_h} \). We also introduce

\[
\phi_j = \begin{pmatrix} c & c^\dagger \end{pmatrix}. \tag{6.4}
\]

Now it should be clear that

\[
\psi_j^\dagger \psi_m + \psi_m \psi_j^\dagger = \delta_{ij}. \tag{6.5}
\]

Indeed, if \( m = j \leq N_h \), this amounts to anticommutations between \( a_m \) and \( a_j^\dagger \). If \( m > N_h \), \( n > N_h \), this is again an anticommutator between \( a_{m-N_h}^\dagger \), \( a_{n-N_h} \). If \( m > N_h \) and \( j < N_h \), this is an anticommutator between \( a_{m-N_h}^\dagger \) and \( a_j^\dagger \), which is zero. Finally, if \( m \leq N_h \), \( j > N_h \) then this is an anticommutator between \( a_m \) and \( a_{j-N_h} \), which is also zero.

In the same way,

\[
\phi_j^\dagger \phi_m + \phi_m \phi_j^\dagger = \delta_{mj}. \tag{6.6}
\]

Now let us use that

\[
\phi_j = \sum_m U_{jm} \psi_m, \quad \phi_j^\dagger = \sum_m \psi_m^\dagger U_{mj}. \tag{6.7}
\]

Substitute these into (6.6), we find

\[
\sum_{nl} \left[ \psi_n^\dagger U_{nj}^\dagger U_{ml} \psi_l + U_{ml} \psi_l^\dagger \psi_n^\dagger U_{nj}^\dagger \right] = \delta_{mj}. \tag{6.8}
\]

Now we note that

\[
\psi_n^\dagger \psi_l + \psi_l \psi_n^\dagger = \delta_{nl}. \tag{6.9}
\]

This gives

\[
\sum_n U_{mn} U_{nj}^\dagger = \delta_{mj}. \tag{6.10}
\]

This means \( U U^\dagger = I \), or \( U \) is a unitary matrix.

Let us write the Hamiltonian (4.8) in the following way

\[
H = (\begin{pmatrix} a & a^\dagger \end{pmatrix}) \begin{pmatrix} \gamma & \frac{\beta}{2} \Delta \\ -\frac{\beta}{2} \Gamma & -\gamma - \frac{\beta}{2} \Delta \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} = \psi^\dagger \mathcal{H} \psi. \tag{6.11}
\]

Here \( \mathcal{H} \) is a \( 2N_h \) by \( 2N_h \) matrix. We can now employ the Bogoliubov transformation, to write

\[
\psi = U^\dagger \phi. \tag{6.12}
\]

Then

\[
H = \phi^\dagger U \mathcal{H} U^\dagger \phi. \tag{6.13}
\]
We choose $U$ so that $UH^{\dagger}$ is diagonal. Then in terms of $c$ the Hamiltonian becomes
\[
H = H = \sum_{k} h_k \left[ c_k^{\dagger}c_k - c_k c_k^{\dagger} \right],
\]
(6.14)

Here $h_k$ are eigenvalues of $\mathcal{H}$.

So the program is then to find the eigenvalues of $\mathcal{H}$. We write
\[
\mathcal{H} = \left( \gamma + \frac{\beta}{2} \Delta \right) \Sigma_3 + i \frac{\beta}{2} \Gamma \Sigma_2,
\]
(6.15)

where $\Delta$ and $\Gamma$ are matrices such that
\[
\Delta_{xy} = \delta_{x,y+1} + \delta_{x,y-1}, \quad \Gamma_{xy} = \delta_{x,y-1} - \delta_{x,y+1},
\]
(6.16)

and $\Sigma_3$, $\Sigma_2$ are Pauli matrices.

To find $h_k$, we will use a series of unitary transformations.

We first rotate $\Sigma_3$ into $\Sigma_1$ by employing $U = \frac{1}{\sqrt{2}} (1 + i \Sigma_2)$ and $H' = U^\dagger H U$. We find
\[
H = \left( \gamma + \frac{\beta}{2} \Delta \right) \Sigma_1 + i \frac{\beta}{2} \Gamma \Sigma_2 = \left( \begin{array}{cc} \gamma + \frac{\beta}{2} \Delta & 0 \\ 0 & \gamma + \frac{\beta}{2} (\Delta - \Gamma) \end{array} \right).
\]
(6.17)

The eigenvalue equations become
\[
hs_x = \gamma t_x + \beta t_{x+1},
\]
(6.18)
\[
ht_x = \gamma s_x + \beta s_{x-1}.
\]
(6.19)

These can be solved in terms of plane waves,
\[
s_x = s_k e^{ikx}, \quad t_x = t_k e^{ikx}.
\]
(6.20)

Here $x$ varies from 1 to $N_h$. $k$ must vary from 0 to $2\pi$ because $x$ is integer (if $k > 2\pi$, then $e^{ik} = e^{i(k-2\pi)}$, so $k$ larger than $2\pi$ is equivalent to $k-2\pi$).
\[
k = 0, \frac{2\pi}{N_h}, \frac{4\pi}{N_h}, \ldots, 2\pi - \frac{2\pi}{N_h}.
\]
(6.21)

We now find
\[
h s_k = \left( \gamma + \beta e^{ik} \right) t_k, \quad h t_k = \gamma s_k + \beta e^{-ik} s_k.
\]
(6.22)

This is equivalent to finding the eigenvalues of
\[
\left( \begin{array}{cc} 0 & \gamma + \beta e^{ik} \\ \gamma + \beta e^{-ik} & 0 \end{array} \right).
\]
(6.23)
The eigenvalues are then

$$h^2 = (\gamma + \beta e^{i k}) (\gamma + \beta e^{-i k}) = \gamma^2 + 2\gamma \beta (\cos(k) + 1)$$

(6.24)

$$h(k) = \pm \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \cos^2 \left(\frac{k}{2}\right)}.$$  

(6.25)

It is convenient to introduce $q = \pi - k$. As $k$ varies from 0 to $2\pi$, $q$ varies from $-\pi$ to $\pi$.

Then

$$h(q) = \pm \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \sin^2 \left(\frac{q}{2}\right)}.$$ 

(6.26)

Thus the Hamiltonian becomes

$$H = \sum_q \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \sin^2 \left(\frac{q}{2}\right)} \left( c_q^\dagger c_q - c_q c_q^\dagger \right).$$ 

(6.27)

Now we can construct the ground and excited states. If we fill all the states with fermions, then

$$c_q^\dagger c_q - c_q c_q^\dagger = -1.$$ 

(6.28)

This gives the minimal possible energy, that is the ground state. The ground state energy is thus

$$E_0 = \sum_q \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \sin^2 \left(\frac{q}{2}\right)} = N_h \int_{-\pi}^{\pi} dq \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \sin^2 \left(\frac{q}{2}\right)}.$$ 

(6.29)

To create an excitation, we take a fermion with momentum $q$ and remove it. Then for this $q$,

$$c_q^\dagger c_q - c_q c_q^\dagger = 1.$$ 

(6.30)

This increases energy by

$$E_q - E_0 = 2 \sqrt{(\gamma - \beta)^2 + 4\gamma \beta \sin^2 \left(\frac{q}{2}\right)}.$$ 

(6.31)

This gives excitation energy. It’s also possible to remove more than one fermion, to get even higher energy.

Interesting to note that at small $q$ it has a relativistic form,

$$h^2(q) = (\gamma - \beta)^2 + \gamma \beta q^2.$$ 

(6.32)

This is the spectrum of particles of mass $m = |\gamma - \beta|$. 

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7 Critical exponent

The lowest excited state is at $q = 0$, giving

$$E_{q=0} - E_0 = 2 |\gamma - \beta| .$$  \hspace{1cm} (7.1)

This is the gap in the spectrum. The correlation length is the inverse gap and is given by

$$\xi = \frac{1}{2|\gamma - \beta|} .$$  \hspace{1cm} (7.2)

It diverges at $\gamma = \beta$. It is customary, if the correlation length is divergent at some critical $T_c$, to denote the power of its divergence as $\nu$:

$$\xi \propto \frac{1}{|T - T_c|^\nu} .$$  \hspace{1cm} (7.3)

We see that for the 2D Ising model $\nu = 1$. Indeed, recall that

$$\beta = K_h = \frac{J_h}{T}, \quad \gamma = e^{-2K_v} = e^{-\frac{2J_v}{T}} .$$  \hspace{1cm} (7.4)

So

$$\beta - \gamma = \frac{J_h}{T} - e^{-\frac{2J_v}{T}} .$$  \hspace{1cm} (7.5)

The critical temperature $T_c$ is when

$$\frac{J_h}{T_c} - e^{-\frac{2J_v}{T_c}} = 0 .$$  \hspace{1cm} (7.6)

This is when the gap vanishes and the correlation length is infinity. If $T$ is close to $T_c$, then

$$\beta - \gamma \approx \left( -\frac{J_h}{T_c^2} + \frac{2J_v}{T_c} e^{-\frac{2J_v}{T_c}} \right) (T - T_c) = \alpha (T - T_c)$$  \hspace{1cm} (7.7)

with some slope $\alpha$. So

$$\xi \sim \frac{1}{|T - T_c|} .$$  \hspace{1cm} (7.8)

$\nu = 1$ is one of Onsager’s famous exact results.

Another result for the free energy easily follows as well. The free energy is given by

$$F = -T \log Z .$$  \hspace{1cm} (7.9)

Since $Z$ is proportional to $\exp (-N_v E_0)$ where $E_0$ is the ground state energy and $N_v$ is the length of the Ising model in the ”time” direction, that gives

$$F = TN_v E_0 .$$  \hspace{1cm} (7.10)
In other words, the ground state energy in quantum mechanics is the same as the free energy in statistical mechanics. In this problem

\[ E_0 \propto -N_h \int_{-\pi}^{\pi} \frac{dq}{2\pi} \sqrt{(\gamma - \beta)^2 + 4\gamma\beta \sin^2 \left(\frac{q}{2}\right)} \]  

(7.11)

So

\[ F = -TN_vN_h \int_{-\pi}^{\pi} \frac{dq}{2\pi} \sqrt{(\gamma - \beta)^2 + 4\gamma\beta \sin^2 \left(\frac{q}{2}\right)}. \]  

(7.12)

Notice that \( F \) is proportional to \( N_vN_h \), which is the total number of sites in our 2D Ising model, which is like a volume of the system. So \( F \) is extensive (proportional to volume), as we should have expected. The integral in the expression for \( F \) is a known integral, whose value is related to elliptic functions. We are not interested in its exact value. Rather we are only interested in the specific heat

\[ c = -T \frac{\partial^2 F}{\partial T^2} \]  

(7.13)

and only at \( T \) close to \( T_c \). To find it we need to differentiate \( F \) with respect to \( T \), remembering that \( \gamma \) and \( \beta \) are both temperature dependent. We recall that at \( T \) close to \( T_c \), \( \beta - \gamma \sim \alpha (T - T_c) \). So we can write

\[ F = -TN_vN_h \int_{-\pi}^{\pi} \frac{dq}{2\pi} \sqrt{\alpha^2(T - T_c)^2 + 4\gamma(T)\beta(T) \sin^2 \left(\frac{q}{2}\right)}. \]  

(7.14)

Differentiating with respect to \( T \) once gives

\[ \frac{\partial F}{\partial T} = -TN_vN_h \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{\alpha^2(T - T_c) + 2 \frac{\partial}{\partial T} (\gamma\beta) \sin^2 \left(\frac{q}{2}\right)}{\sqrt{\alpha^2(T - T_c)^2 + 4\gamma(T)\beta(T) \sin^2 \left(\frac{q}{2}\right)}} - \]

\[ N_vN_h \int_{-\pi}^{\pi} \frac{dq}{2\pi} \sqrt{\alpha^2(T - T_c)^2 + 4\gamma(T)\beta(T) \sin^2 \left(\frac{q}{2}\right)}. \]  

(7.15)

Computing the second derivative gives

\[ \frac{\partial^2 F}{\partial T^2} = \frac{TN_vN_h}{2} \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{(\alpha^2(T - T_c) + 2 \frac{\partial}{\partial T} (\gamma\beta) \sin^2 \left(\frac{q}{2}\right))^2}{\alpha^2(T - T_c)^2 + 4\gamma(T)\beta(T) \sin^2 \left(\frac{q}{2}\right)} - \]

\[ TN_vN_h \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{\alpha^2 + 2 \frac{\partial^2}{\partial T^2} (\gamma\beta) \sin^2 \left(\frac{q}{2}\right)}{\sqrt{\alpha^2(T - T_c)^2 + 4\gamma(T)\beta(T) \sin^2 \left(\frac{q}{2}\right)}} - \]

\[ N_vN_h \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{\alpha^2(T - T_c) + 2 \frac{\partial}{\partial T} (\gamma\beta) \sin^2 \left(\frac{q}{2}\right)}{\sqrt{\alpha^2(T - T_c)^2 + 4\gamma(T)\beta(T) \sin^2 \left(\frac{q}{2}\right)}}. \]  

(7.16)
As $T \to T_c$, the first and the third term remain finite. The second term can be estimated to behave as $\log |T - T_c|$. So we find

$$c \propto \log \left( \frac{1}{|T - T_c|} \right). \quad (7.17)$$

This is another of Onsager’s famous results. A standard notation in statistical mechanics is

$$c \propto \frac{1}{|T - T_c|^\alpha} \quad (7.18)$$

From here it follows that $\alpha = 0$. 