Nondegenerate and degenerate limits for ideal Fermi gases. The pressure and number density and energy density for an ideal Fermi gas are given by

\[ P(\mu, T) = kT \int a(\varepsilon) \ln \left(1 + e^{-\beta(\varepsilon - \mu)}\right) d\varepsilon \]
\[ n(\mu, T) = \int a(\varepsilon) \frac{1}{e^{\beta(\varepsilon - \mu)} + 1} d\varepsilon \]
\[ u(\mu, T) = \int a(\varepsilon) \frac{\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1} d\varepsilon \]

(a) Using the energy density of states for free spin-\(1/2\) fermions

\[ a(\varepsilon) = \frac{\sqrt{2m^3\varepsilon}}{\pi^2\hbar^3}, \]

show that when \(\beta\mu \ll -1\) the density and pressure take on the forms we derived for a classical ideal gas (other than a spin degeneracy factor in the density):

\[ n = \frac{2e^{\beta\mu}}{\lambda^3}, \]
\[ P = nkT. \]

Solution

For large and negative \(\mu\) we can treat \(e^{-\beta(\varepsilon - \mu)} \ll 1\) and \(e^{+\beta(\varepsilon - \mu)} \gg 1\) for all \(\varepsilon\) so

\[ P(\mu, T) \approx kT \int \frac{\sqrt{2m^3\varepsilon}}{\pi^2\hbar^3} e^{-\beta(\varepsilon - \mu)} d\varepsilon = nkT \quad \text{where} \]
\[ n(\mu, T) \approx \int \frac{\sqrt{2m^3\varepsilon}}{\pi^2\hbar^3} e^{-\beta(\varepsilon - \mu)} d\varepsilon = 2\frac{e^{\beta\mu}}{\lambda^3} \quad \text{and}, \]
\[ \lambda = \frac{\hbar}{\sqrt{2\pi mkT}}, \]

(b) Now, allowing the density of states \(a(\varepsilon)\) to be an arbitrary function, calculate the leading temperature dependence of the specific heat at constant chemical potential at low temperature and show

\[ c_\mu = T \left( \frac{\partial s}{\partial T} \right)_\mu = T \left( \frac{\partial^2 P}{\partial T^2} \right)_\mu = \frac{\pi^2}{3} k^2 T a(\varepsilon_F) + \cdots. \]
This implies that the measurement of the linear dependence of the specific heat on temperature gives a direct measure of the density of states at the Fermi energy.

Solution
Taking two temperature derivatives of the pressure gives the specific heat as

\[ c_\mu = T \left( \frac{\partial s}{\partial T} \right)_\mu = T \left( \frac{\partial^2 P}{\partial T^2} \right)_\mu = \frac{k}{(kT)^2} \int a(\varepsilon)(\varepsilon - \mu)^2 \frac{e^{\beta(\varepsilon - \mu)}}{(e^{\beta(\varepsilon - \mu)} + 1)^2} d\varepsilon. \]

Expand the density of states in a Taylor series centered at \( \mu \) and let \( x = \beta(\varepsilon - \mu) \). The even order terms in the expansion, starting with constant term, will give nonzero results by symmetry.

\[ c_\mu = \frac{k}{(kT)^2} \int \left[ a(\mu) + a'(\mu)(\varepsilon - \mu) + \frac{1}{2} a''(\mu)(\varepsilon - \mu)^2 + \cdots \right] (\varepsilon - \mu)^2 \frac{e^{\beta(\varepsilon - \mu)}}{(e^{\beta(\varepsilon - \mu)} + 1)^2} d\varepsilon, \]

\[ = k(kT)a(\mu) \int x^2 \frac{e^x}{(e^x + 1)^2} dx + k(kT)^3 a''(\mu) \int x^4 \frac{e^x}{(e^x + 1)^2} dx \]

\[ = \frac{\pi^2 k(kT)a(\mu)}{3} + \frac{7\pi^4 k(kT)^3 a''(\mu)}{15} + \cdots \]