1. The occupancy of the first excited state for temperatures below the condensation temperature is given by

\[ \langle N_1 \rangle = \frac{1}{e^{\beta \hbar \omega_0} - 1} \approx \frac{kT}{\hbar \omega_0} \sim N^{1/3} \left( \frac{T}{T_c} \right) \] since \( kT_c \gg \hbar \omega_0 \).

\[ \frac{\langle N_1 \rangle}{N} \sim \frac{1}{N^{2/3}} \left( \frac{T}{T_c} \right) \to 0 \] in the thermodynamic limit.

2. Pathria and Beale 7.19. The density of states for a two-dimensional harmonic oscillator is

\[ a(\varepsilon) = \frac{\varepsilon}{\hbar \omega_0} \] so the number particles in the trap is given by

\[ N(T, \mu) = \int d\varepsilon \frac{\varepsilon}{\hbar \omega_0^2} e^{\beta(\varepsilon - \mu)} - 1 \]

As \( T \to T_c, \mu \to 0 \) so

\[ N = \int d\varepsilon \frac{\varepsilon}{(\hbar \omega_0)^2} e^{\beta(\varepsilon)} - 1 = \left( \frac{kT_e}{\hbar \omega_0} \right)^2 \int \frac{xdx}{e^x - 1} = \zeta(2) \left( \frac{kT_e}{\hbar \omega_0} \right)^2 = \frac{\pi^2}{6} \left( \frac{kT_e}{\hbar \omega_0} \right)^2. \]

so \( kT_e = \hbar \omega_0 \sqrt{6N/\pi^2} \). The condensate fraction for \( T \leq T_c \) is

\[ N_0/N = 1 - (T/T_c)^2. \] For this two-dimensional theory to be valid, the occupancy of the first excited z-state must be negligible which requires \( \hbar \omega_z \gg kT_c \sim \sqrt{N \hbar \omega_0} \), i.e. \( \omega_z \gg \sqrt{N} \omega_0 \).

3. Solution: for power-law density of states

\[ a(\varepsilon) = \alpha \frac{\epsilon^\sigma}{(d-1)!\epsilon_1^{\sigma+1}} \]

\[ \langle N \rangle = \left( \frac{kT}{\epsilon_1} \right)^{\sigma+1} \sum_{j=1}^{\infty} \frac{Z^j}{j^{\sigma+1}} \]

\[ \langle N \rangle = \alpha \left( \frac{kT}{\hbar \omega_0} \right)^{\sigma+1} g_{\sigma+1}(z) \]

\[ N = \alpha \left( \frac{kT}{\hbar \omega_0} \right)^{\sigma+1} \zeta_{\sigma+1} \]

\[ \zeta(\sigma + 1) = \sum_{j=1}^{\infty} \frac{1}{j^{\sigma+1}} \approx \int_1^{\infty} \frac{ds}{s^{\sigma+1}} = \frac{1}{\sigma} \] for \( \sigma \ll 1 \).

\[ kT_e/\epsilon_1 = \left( \frac{N}{\alpha \zeta_{\sigma+1}} \right)^{1/(\sigma+1)} \approx \left( \frac{\sigma N}{\alpha} \right)^{1/(\sigma+1)} \to 0 \] as \( \sigma \to 0 \).
For \(d\)-dimensional harmonic oscillator,

\[
a(\varepsilon) = \frac{\varepsilon^{d-1}}{(d-1)! (\hbar \omega_0)^d}
\]

\[
\langle N \rangle = \int_0^\infty a(\varepsilon) d\varepsilon = \left(\frac{kT}{\hbar \omega_0}\right)^d \sum_{j=1}^\infty \frac{\varepsilon^d}{j^d}
\]

\[
\langle N \rangle = \left(\frac{kT}{\hbar \omega_0}\right)^d \varphi_d(z)
\]

\[
N = \left(\frac{kT_c}{\hbar \omega_0}\right)^d \zeta(d)
\]

\[
\zeta(d) = \sum_{j=1}^\infty \frac{1}{j^d} \approx \int_1^\infty \frac{ds}{s^d} = \frac{1}{d-1} \quad \text{for} \quad d-1 \ll 1.
\]

\[
kT_c/\hbar \omega_0 = \left(\frac{N}{\zeta(d)}\right)^{1/d} \approx (d-1)N^{1/d} \to 0 \quad \text{as} \quad d \to 1.
\]

The ground state occupancy and internal energy below the critical temperature are given by

\[
\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^d,
\]

\[
U(T) = \frac{d\zeta(d+1)}{(d-1)!} \left(\frac{kT}{\hbar \omega_0}\right)^{d+1}, \text{i.e.}
\]

\[
\frac{U(T)}{NkT_c} = \frac{d\zeta(d+1)}{\zeta(d)} \left(\frac{T}{T_c}\right)^{d+1} \approx (d-1)\zeta(2) \left(\frac{T}{T_c}\right) \quad \text{for} \quad d-1 \ll 1.
\]

4. Pathria and Beale 7.21 Using expressions (7.2.12) and (7.2.23), we readily get

\[
\frac{U}{N} = \frac{\pi^4}{30\zeta(3)} kT \approx 2.7 kT.
\]

Note that the numerical factor appearing here is actually \(\Gamma(4)\zeta(4)/\Gamma(3)\zeta(3)\).

5. Pathria and Beale 7.23 The total power radiated by the sun (radius \(R_s\) and temperature \(T_s\) is

\[
P_s = 4\pi \sigma T_s^4 R_s^2
\]

where \(\sigma = \pi^2 k^4 T^4 / 60 \hbar^3 c^5\), so the intensity is the earth’s distance from the sun \(R_e\) is

\[
I_e = \frac{4\pi \sigma T_s^4 R_s^2}{4\pi R_e^2} = \sigma \left(\frac{R_s}{R_e}\right)^2.
\]
The radiation pressure acting on a blackbody is \( P = I/c \). The equilibrium temperature of a flat plate blackbody absorber (one-sided) is determined by the power absorbed being equal to the power emitted, i.e. \( \sigma T_e^4 = I_e \) so \( T_e = T_s \sqrt{R_s/R_e} \approx 396 \text{K} \). This, not coincidentally, is about the equilibrium temperature of the earth. A better estimate would include the emissivity of the sun and earth, the albedo of the earth, and the spherical shape of the earth and its rotation which averages the temperature over the whole earth.

6. Pathria and Beale 7.24 The number density of photons in the cosmic microwave background (CMB) follows from equation (7.3.23)

\[
\frac{\zeta(3)}{\pi^2} \left( \frac{kT}{\hbar c} \right)^3 \approx 4.10 \times 10^8 \text{m}^{-3} \approx 410 \text{cm}^{-3}
\]

The energy density is

\[
u = \frac{\pi^2}{15} \left( \frac{kT}{\hbar c} \right)^4 \approx 4.17 \times 10^{-14} \text{J/m}^3.
\]

The entropy density is

\[
s = \frac{4\pi^2 k}{45} \left( \frac{kT}{\hbar c} \right)^3 \approx 1.48 \times 10^9 \text{K m}^{-3} \approx 2.04 \times 10^{-14} \text{J/m}^3 \text{K}.
\]