1. (15%) Prove that the secondary maxima of a single-slit diffraction pattern occur when \( \beta = \tan \beta \).

2. (15%) A single-slit Fraunhofer diffraction pattern is formed using white light. For what wavelength does the second minimum coincide with the third minimum produced by a wavelength of \( \lambda = 400 \text{nm} \).

3. (20%) For an apodized slit which has a transmission function of

\[
 f(y) = \frac{1}{a} \sqrt{\frac{2}{\pi}} \exp \left[ -2 \left( \frac{y}{a} \right)^2 \right],
\]

proof that the Fraunhofer diffraction pattern is

\[
 E \propto \int_{-\infty}^{\infty} f(y) e^{i k y} dy = \exp \left(-\frac{k^2 a^2}{8} \right)
\]

by using the relation \( \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi} \).

Note that the diffraction pattern has no sidebands.

4. (15%) Consider a double slits aperture, each slit has a width \( b \), and are separated by a distance \( h \). Prove that the intensity of the Fraunhofer diffraction pattern is

\[
 I = I_0 \left( \frac{\sin \beta}{\beta} \right)^2 \cos^2 \gamma
\]

where \( \beta = \frac{1}{2} k b \sin \theta \) and \( \gamma = \frac{1}{2} k h \sin \theta \).

5. (15%) Continue on problem #4, the first term of the diffraction pattern of a double slits aperture \( \left( \frac{\sin \beta}{\beta} \right)^2 \) typically accounts for diffraction pattern of a single slit, and the \( \cos^2 \gamma \) terms corresponds to the interference between the two slits. Missing orders occur at those values of \( \sin \theta \) which satisfy, at the same time, the condition for interference maxima and the condition for diffraction minima. Show that this leads to the condition \( \left( \frac{b}{b} \right) = \text{integer} \).

6. (20%) Plot the diffraction pattern \( I(\theta) \) of the double slits aperture for \( \theta \) from -180° to 180° in Problem #5, where \( I_0 = 1, b = 1 \mu m, h = 3 \mu m \) and the optical wavelength \( \lambda = 632nm \).
HW #5 Solution

For a single slit Fraunhofer Diffraction pattern,

\[ I = I_0 \left( \frac{\sin \beta}{\beta} \right)^2 \]

to find the maximum, set \( \frac{dI}{d\beta} = 0 \)

\[ \frac{dI}{d\beta} = I_0 \cdot 2 \left( \frac{\sin \beta}{\beta} \right) \frac{d}{d\beta} \left( \frac{\sin \beta}{\beta} \right) \]

\[ = I_0 \cdot \frac{2 \sin^2 \beta}{\beta^2} \left[ -\frac{1}{\beta^2} \sin \beta + \frac{1}{\beta} \cos \beta \right] \]

\[ = I_0 \cdot \frac{2 \sin \beta}{\beta} \left[ \beta \cos \beta - \sin \beta \right] \]

\[ = 0 \]

\[ \therefore \beta \cos \beta = \sin \beta \]

\[ \Rightarrow \beta = \tan \beta \]
2. For single slit diffraction, the intensity
\[ I = I_0 \left( \frac{\sin \beta}{\beta} \right)^2 \]
the minima occur when \( \beta = m\pi \), \( m = \pm 1, \pm 2, \ldots \)

Since \( \beta = \frac{1}{2} kb \sin \theta \)
\[ = \frac{\pi}{\lambda} b \sin \theta \]
At minima, \( \beta = \frac{\pi}{\lambda} b \sin \theta = m\pi \)
\[ \Rightarrow b \sin \theta = m\lambda \]
therefore \( b \sin \theta = 3\lambda \) \( \text{where}\ \lambda = 400\text{nm} \)
and \( b \sin \theta = 2\lambda' \)

\[ \Rightarrow 3\lambda = 2\lambda' \]
\[ \Rightarrow \lambda' = \frac{3}{2} \lambda \]
\[ = 600\text{nm} \]

For Fraunhofer diffraction,

\[ E = \int_{-\infty}^{\infty} f(x) e^{-i k y} \, dy \]

\[ = \int_{-\infty}^{\infty} \frac{1}{a} \sqrt{\frac{2}{\pi}} e^{-2\left(\frac{y^2}{a^2}\right)} e^{-i k y} \, dy \]

\[ = \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{5y^2}{a^2}\right)} - i k y \, dy \]

Since \((\frac{5y^2}{a^2}) + i k y = (\frac{5y^2}{a^2}) + i k y + \left(\frac{ik a}{2\sqrt{2}}\right)^2 - \left(\frac{ik a}{2\sqrt{2}}\right)^2\)

\[ = \left[\frac{5y^2}{a^2} + \frac{ik a}{2\sqrt{2}}\right]^2 + \frac{k^2 a^2}{8} \]

\[ E \propto \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{5y^2}{a^2} + \frac{ik a}{2\sqrt{2}}\right)^2} - \frac{k^2 a^2}{8} \, dy \]

\[ = \frac{1}{a} \sqrt{\frac{2}{\pi}} e^{-\frac{k a^2}{8}} \int_{-\infty}^{\infty} e^{-\left(\frac{5y^2}{a^2} + \frac{ik a}{2\sqrt{2}}\right)^2} \, dy \]

\[ = \frac{1}{a} \sqrt{\frac{2}{\pi}} e^{-\frac{k a^2}{8}} \sqrt{\frac{2}{\pi}} e^{-\left(\frac{5y^2}{a^2} + \frac{ik a}{2\sqrt{2}}\right)^2} \, dy \]

Using the relation

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \]

\[ E \propto e^{-k^2 a^2 / 8} \]
4. The Fraunhofer diffraction pattern is

\[ E \propto \int_{\text{double slit}} e^{i ky \sin \theta} \, dy \]

\[ = \int_0^b e^{i ky \sin \theta} \, dy + \int_h^{h+b} e^{i ky \sin \theta} \, dy \]

\[ = \frac{1}{ik \sin \theta} \left[ \int_0^b e^{d(i ky \sin \theta)} \right. \\
\left. + \int_h^{h+b} e^{iky \sin \theta} \right] \left( d(i ky \sin \theta) \right) \]

\[ = \frac{1}{ik \sin \theta} \left[ e^{ikb \sin \theta} e^{ik(\frac{b}{2}) \sin \theta} - e^{ikh \sin \theta} \right] \]

\[ = \frac{4 \lambda}{ik \sin \theta} \left( -e^{-i k(b/2) \sin \theta} + e^{-i k(h/2) \sin \theta} \right) \left( \frac{e^{i k(h/2) \sin \theta} - e^{-i k(h/2) \sin \theta}}{2i} \right) \]

\[ = \frac{2b}{k \sin \theta} \left( e^{i \frac{1}{2} \kappa b \sin \theta} \cdot e^{i \frac{1}{2} \kappa h \sin \theta} \cdot \sin \left( \frac{1}{2} \kappa b \sin \theta \right) \cos \left( \frac{1}{2} \kappa h \sin \theta \right) \right) \]

Setting \( \frac{1}{2} \kappa b \sin \theta = \beta \) and \( \frac{1}{2} \kappa h \sin \theta = \sigma \)

\[ \Rightarrow E \propto 2b e^{i \beta} e^{i \sigma} \frac{\sin \beta}{\beta} \cos \sigma \]

Therefore, the intensity

\[ I = I_0 \left( \frac{\sin \beta}{\beta} \right)^2 \cos^2 \sigma \]
5. For Fraunhofer diffraction due to a double slit, the intensity is

\[ I(\theta) = I_0 \frac{\sin^2 \beta}{\beta^2} \cos^2 \gamma \]

where \( \beta = \frac{1}{2} K b \sin \theta \) and \( \gamma = \frac{1}{2} K h \sin \theta \)

Missing order occurs when

 interfere\ence maxima \( \Rightarrow \) \( \max \{\cos^2 \gamma\} \)

\[ \gamma = m\pi \quad m = 0, \pm 1, \pm 2 \ldots \]

diffraction minimum \( \Rightarrow \) \( \min \left( \frac{\sin^2 \beta}{\beta^2} \right) \)

\[ \beta = n\pi \quad n = \pm 1, \pm 2 \ldots \]

\[ \Rightarrow \quad \frac{\gamma}{\beta} = \frac{m\pi}{n\pi} \]

or \( \frac{h}{b} = \frac{m}{n} = \text{integer} \)
% Problem #6
% plotting the double slit diffraction pattern

clear all;

theta_angle = -180:.01:180;
theta = theta_angle/180*pi;

lambda = 632e-9;
b=1e-6;
h=3e-6;

k=2*pi/lambda

beta = .5*k*b*sin(theta);
gamma = .5*k*h*sin(theta);

I=(sin(beta)./beta.*cos(gamma)).^2;

plot(theta_angle, I);
xlabel('Angle');
ylabel('Diffraction Pattern');