1. a. (10%) A 5 cm optical crystal with a refractive index of \( n = (1.5 + 10\sqrt{z}) \), where \( z \) is in meter. Calculate the optical path length of the crystal.
b. (10%) Calculate the length of another crystal to have the same optical path length with an refractive index of \( n = (1.5 + 10z) \).
c. (5%) Calculate the propagation time for the two crystals assuming the pulse is infinitely short.

2. a. (15%) Show by means of the Jones calculus that circularly polarized light can be produced by sending an arbitrary light \( \begin{bmatrix} A \\ Be^{-\Delta} \end{bmatrix} \), where \( \Delta \) is an arbitrary phase, through a linear polarizer and a quarter-wave plate only in the right arrangement of these two optics. The Jones matrix for the quarter-wave plate is \( \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \), and the linear polarizer is \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \).
b. (10%) Verify that a circular polarizer whose Jones matrix is \( \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \) is completely transparent to one type of circularly polarized light \( \begin{bmatrix} 1 \\ -i \end{bmatrix} \) and opaque to the opposite circular polarization \( \begin{bmatrix} 1 \\ i \end{bmatrix} \).

3. An anti-reflection coating is often used to minimize optical reflection from a glass substrate, as shown in the graph below. In normal incident,
a. (5%) What is the Fresnel field reflection coefficient \( r_1 \) for the first interface?
b. (5%) What is the Fresnel field reflection coefficient \( r_2 \) for the second interface? (assume reflections on interfaces are negligible \( i \approx 1 \))
c. (15%) In order to cancel these two reflections, the field reflections have to be equal \( (r_1 = r_2) \), proof that \( n = \sqrt{n'} \) in this situation.
4. (10%) The numerical aperture of a waveguide is defined as \( NA = (n_2^2 - n_1^2)^{1/2} \). When \( n_1 \approx n_2 \), show that the NA can be approximated by
   \[ NA \approx n_2 (2\Delta)^{1/2}, \quad \Delta = (n_2 - n_1)/n_2 \]

b. (15%) Assume the waveguide has a length of \( L \), and the core and the cladding refractive indexes are \( n_2 \) and \( n_1 \), respectively. A rough estimation is sometimes used to determine the modal dispersion of a multimode waveguide. Show that the propagation time difference \( dT \) between the ray traveling straight along the waveguide (\( \theta_i = 0 \)) and the ray input to the waveguide at the acceptance angle of the numerical aperture (\( \theta_i = \theta_a \)) is
   \[ dT = \frac{L n_2^2}{c \ n_1} \Delta \]
Physics 4510  Optics  Quiz #1

1) a) for \( n = (1.5 + 10z) \)
\[
\text{OPL} = \int_0^\ell \left(1.5 + 10 \frac{z}{2}\right) \, dz
\]
\[
= \left[1.5z + 10 \left(\frac{z}{3}\right)z^{3/2}\right]_0^\ell
\]
\[
= 1.5\ell + \left(\frac{20}{3}\right)\ell^{3/2}
\]
when \( \ell = 0.05 \, m \)
\[
\text{OPL} = 1.5(0.05) + \left(\frac{20}{3}\right)(0.05)^{3/2}
\]
\[
= 0.15 \, m = 15 \, cm
\]

b) for \( n = (1.5 + 10z) \)
\[
\text{OPL} = \int_0^\ell \left(1.5 + 10z\right) \, dz
\]
\[
= \left[1.5z + 10\left(\frac{1}{5}\right)z^2\right]_0^\ell
\]
\[
= 1.5\ell + 5\ell^2
\]
if \( \text{OPL} = 0.15 \)
\[
\Rightarrow 5\ell^2 + 1.5\ell - 0.15 = 0
\]
\[
\Rightarrow \ell = 0.079 \, m \approx 8 \, cm
\]

C) Since the OPLs of both crystals are the same, the propagation time is also the same,
\[
\Rightarrow t = \frac{\text{OPL}}{c} = \frac{0.15}{3 \times 10^8} = 5 \times 10^{-10} \, s
\]
\[
= 0.5 \, ns
\]
2) a) Since the Jones vector for a linear polarizer is \[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]
and the Jones vector for an arbitrary light is \[ \begin{bmatrix} A \\ Be^{i\alpha} \end{bmatrix} \]
therefore, if light goes through first the polarizer, then \(1/4\) wp

\[
\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ Be^{i\alpha} \end{bmatrix} = \begin{bmatrix} A \\ iA \end{bmatrix} = A \begin{bmatrix} 1 \\ i \end{bmatrix}
\]
produces circularly polarized light

otherwise,

\[
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} A \\ Be^{i\alpha} \end{bmatrix} = \begin{bmatrix} A \\ iBe^{i\alpha} \end{bmatrix} = \begin{bmatrix} A + iBe^{i\alpha} \\ 0 \end{bmatrix} = \begin{bmatrix} A + Be^{i(\alpha + \pi/2)} \\ 0 \end{bmatrix}
\]
produces linearly polarized light.
2 b) \[
\begin{bmatrix}
-i & i \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
-i \\
1
\end{bmatrix} = \begin{bmatrix}
1+i \\
-i + i
\end{bmatrix} = 2\begin{bmatrix}
-i
\end{bmatrix}
\]

\Rightarrow \text{ completely transparent for } \begin{bmatrix}
-i
\end{bmatrix}

\begin{bmatrix}
-i & i \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
-i \\
1
\end{bmatrix} = \begin{bmatrix}
-1 & -1 \\
-i + i
\end{bmatrix} = \begin{bmatrix}
0
\end{bmatrix}

\Rightarrow \text{ completely opaque}
3) 

a) \[ r_1 = \frac{1-n}{1+n} \]

b) \[ r_2 = \frac{n-n'}{n+n'} \]

c) if \( r_1 = r_2 \)

\[ \frac{1-n}{1+n} = \frac{n-n'}{n+n'} \]

\[ (1-n)(n+n') = (1+n)(n-n') \]

\[ n^2 - n - n' + n' = n + n^2 - n' - nn' \]

\[ \Rightarrow 2n^2 = 2n' \]

\[ \Rightarrow n = \sqrt{n'} \]
a) \[ \text{NA} = \left( n_2^2 - n_1^2 \right)^{1/2} \]
\[ = n_2 \left( 1 - \frac{n_1^2}{n_2^2} \right)^{1/2} \]
\[ = n_2 \left( \frac{n_2^2 - n_1^2}{n_2^2} \right)^{1/2} \]
\[ = n_2 \left[ \frac{(n_2 + n_1)(n_2 - n_1)}{n_2^2} \right]^{1/2} \]

Since \( n_1 \approx n_2 \)
\[ \Rightarrow \text{NA} \approx n_2 \left[ \frac{2n_1(n_2 - n_1)}{n_2^2} \right]^{1/2} \]
\[ = n_2 \left( 2 \Delta \right)^{1/2} \]
where \( \Delta = \frac{n_2 - n_1}{n_2} \)

b)

For the ray going into the fiber with \( \Theta_a \), the optical path length inside the core is
\[ (OPL)_{\Theta_a} = n_2 L \left( \frac{1}{\sin \Theta_c} \right) \]

where \( \Theta_c \) is the critical angle, which
\[ \sin \Theta_c = \frac{n_1}{n_2} \]
\[ \Rightarrow (OPL)_{\Theta_a} = n_2 L \left( \frac{n_2}{n_1} \right) \]
for the ray going straight into the fiber
($\theta_i = 0$)

$(\text{OPL})_o = n_2 L$

therefore, the optical path difference

$d(\text{OPL}) = (\text{OPL})_\theta - (\text{OPL})_o$

$= n_2 L \left( \frac{n_2}{n_1} - 1 \right)$

$= n_2 L \left( \frac{n_2 - n_1}{n_1} \right)$

thus, the propagation time difference

$dT = \frac{d(\text{OPL})}{c}$

$= \frac{n_2}{c} L \left( \frac{n_2 - n_1}{n_1} \right)$

$= \frac{L}{c} \frac{n_2^2}{n_1} \left( \frac{n_2 - n_1}{n_2} \right)$

$= \frac{L}{c} \frac{n_2^2}{n_1} \Delta$