Midterm -I

Phys4410, Spring 2013

Feb 21, 2013

1) Consider the Hamiltonian

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 + \lambda x. \]

a) (10 pts) By considering the term \( \lambda x \) as a perturbation calculate the energy corrections at lowest order for an arbitrary state \( |n> \).

b) (10 pts) Calculate the second order energy correction to the state \( |n = 0> \).

Hint: It is helpful to write \( x = \sqrt{\hbar/(2m\lambda)}(a_+ + a_-) \) with \( a_+|n>=\sqrt{n+1}|n+1> \), \( a_-|n>=\sqrt{n}|n-1> \).

\[ E_n^{(1)} = \langle n | \lambda x | n \rangle = \lambda \langle n | (a_+ + a_-) | n \rangle \sqrt{\frac{\hbar}{2m\omega}} = 0 \]

\[ \sqrt{n+1} |n+1> + \sqrt{n} |n-1> \]

\[ \text{orthogonal states to } |n> \]

b) \[ E_0^{(2)} = \sum_{k \neq 0} \frac{|\langle k | H_1 | 0 \rangle|^2}{E_0 - E_k} = \gamma. \]

\[ H_1 |0> = \lambda \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-) |0> = \lambda \sqrt{\frac{\hbar}{2m\omega}} |1> \]

so only \( k=1 \) term contributes; \( E_0^{(0)} - E_1^{(0)} = -\hbar \omega \)

\[ \gamma = -\lambda^2 \frac{\hbar^2}{2m\omega} \frac{1}{\hbar \omega} = \frac{\lambda^2}{2m\omega^2} \]
2) An electron is in the \( n = 3, j = 3/2 \) excited state of the hydrogen.

a) (5 pts) What are the possible values of the orbital angular momentum in this state?

b) (10 pts) Assume that an experiment determines that the orbital quantum number is \( l = 1 \). Express the \( |j = 3/2, m_j = 3/2, l = 1 > \) state in terms of \( |l, m_l, s, m_s > \) eigenstates.

c) (15 pts) With the same assumption as in b), find \( |j = 3/2, m_j = 1/2, l = 1 > \) in terms of \( |l, m_l, s, m_s > \) eigenstates.

d) (5 pts) If you did c) using a table of Clebsch-Gordon coefficients, repeat c) now by explicitly calculating the necessary coefficients. If you already did the calculation, check the Clebsch-Gordon table to verify it. In any case, circle the relevant column in the table.

e) (5 pts) If the electron is in the \( |j = 3/2, m_j = 1/2, l = 1 > \) state, what is the probability that if the \( z \) component of the spin of this electron is measured, it is \( +\hbar/2 \)?

Hint: \( J_-|j, m >= \sqrt{(j + m)(j - m + 1)}|j, m - 1 > \)

\[ \begin{align*}
\text{a)} & \quad n = 3 : \quad l = 0, 1, 2 \\
\text{b)} & \quad l = 1 \rightarrow j = \frac{3}{2} \text{ or } \frac{1}{2} \rightarrow |j = \frac{3}{2}, m_j = \frac{3}{2}, l = 1 > = |l = 1, m_l = 1, m_s = \frac{1}{2} > \\
\text{c)} & \quad J_- \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} = \sqrt{3} \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \\
& \Rightarrow \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \\
\text{d)} & \quad \text{We need the table} \quad (\frac{1}{2}, \frac{1}{2}) \quad \text{Pick the column} \quad (\frac{3}{2}, \frac{1}{2}) : \\
& \quad \text{Coefficients} : \begin{bmatrix} \sqrt{2}/3 & \sqrt{2}/3 & \sqrt{2}/3 \end{bmatrix} \\
\text{e)} & \quad \langle w_s = \frac{1}{2} | j = \frac{3}{2}, m_j = \frac{3}{2}, l = 1 > = \sqrt{2}/3, \text{ the probability is } \frac{2}{3} \}.\]
Note: A square-root sign is to be understood over every coefficient, e.g., for \(-8/15\) read \(-\sqrt{8/15}\).

\[ Y_1^0 = \sqrt{\frac{3}{8\pi}} \cos \theta \]
\[ Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} \]
\[ Y_2^0 = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \]
\[ Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi} \]
\[ Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi} \]

\[ Y_{\ell}^{-m} = (-1)^m Y_{\ell}^m \]

\[ \langle j_1 j_2 m_1 m_2 j_1 j_2 J M \rangle = (-1)^{J-J_1-J_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 J M \rangle \]
3) Consider the \( n = 2 \) level of the hydrogen \( |2, l, m_l, m_s \rangle \).

a) (5 pts) How many states are there in this level?

b) (15 pts) The atom is placed in a strong magnetic field that is described by the Hamiltonian

\[
H_Z = \frac{e}{2m} B_{ext} (L_z + 2S_z).
\]

If the fine structure can be neglected, how many distinct energy levels does the \( n = 2 \) state splits? Make a table listing clearly the quantum numbers and first order energy correction of each state.

c) (20 pts) Now add the effect of the spin-orbit interaction as a perturbation. For simplicity use the interaction Hamiltonian

\[
H_1 = \lambda \bar{S} \bar{L}.
\]

and calculate the first order energy corrections. If there are degenerate states, be sure to consider the off-diagonal matrix elements. Add the corresponding energy levels to the table you made in part b).

Hint: It helps to use the identity \( \bar{S} \bar{L} = S_+ L_- + S_- L_+ + S_z L_z \).

a) \( n = 2 \):

\[
\begin{array}{c|cccc}
 l & m_e & m_s & (m_e + 2m_s) & \bar{J}S \\
 0 & 0 & \frac{1}{2} & 1 & 0 \\
 0 & 0 & -\frac{1}{2} & 1 & 0 \\
 1 & 1 & \frac{1}{2} & 2 & -\frac{1}{2} \\
 1 & 1 & -\frac{1}{2} & 2 & -\frac{1}{2} \\
 1 & 0 & \frac{1}{2} & 1 & 0 \\
 1 & 0 & -\frac{1}{2} & 1 & 0 \\
 1 & -1 & \frac{1}{2} & 0 & -3 \frac{1}{2} \\
 1 & -1 & -\frac{1}{2} & 2 & \frac{1}{2} \\
\end{array}
\]

\[
H_Z (l m_e m_s) = \frac{e}{2m} B (m_e + 2m_s) \langle l m_e m_s | l m_e m_s \rangle
\]

\[\text{could be } \pm 2: \text{ un-deg.} \quad 0, \pm 1: \text{double degenerate} \]
Quick check: we still have all 8 states:

The energy correction is the exact energy change.

\[ E_{L \omega_L \omega_S} = \frac{e^2}{2 \hbar} (m_e + 2m_\omega) \]

1) We need to do degenerate PT on \((m_e + \omega_S) = 0, \pm 1\)

i) \(m_e + 2\omega_S = 0\):

\[ l = 1, \omega_L = 0, \omega_S = -\frac{1}{2} \quad (l = 0) \quad m_e = -1, \omega_S = \frac{1}{2} \]

Calculate

\[ \overline{S} = S_+ L_+ + S_- L_- + L_2 S_2 \]

\[ = c_1 \left( |1, 0, \frac{1}{2} \rangle + (-1) \left| 1, 1, -\frac{1}{2} \right\rangle \right) \]

\[ \overline{S} \left| 1, 1, -\frac{1}{2} \right\rangle = c_2 \left| 1, 0, -\frac{1}{2} \right\rangle + (1) \left| 1, 1, -\frac{1}{2} \right\rangle \]

Since the \((1, \omega_L, \omega_S)\) states are orthonormal, the

2x2 matrix \(\langle \overline{S} | S \rangle\) is diagonal with

diagonal elements

\[ E_1 = \langle 1, 0, -\frac{1}{2} \rangle \Lambda = -\frac{3}{2} \quad \text{the states are still} \]

\[ E_2 = \langle 1, 1, -\frac{1}{2} \rangle \Lambda = -\frac{1}{2} \quad \text{degenerate} \]

ii) \(m_e + 2\omega_S = 1\):

\[ l = 0, m_e = 0, \omega_S = \frac{1}{2}, l = 1, m_e = 0, \omega_S = \frac{1}{2} \]

Again, \(\overline{S} \left| 0, 0, \frac{1}{2} \right\rangle = c_1 \left| 0, 1, -\frac{1}{2} \right\rangle \]

\[ \overline{S} \left| 1, 1, \frac{1}{2} \right\rangle = c_1 \left| 1, 1, -\frac{1}{2} \right\rangle \]

Now both the diagonal and off-diagonal elements

vanish \(\leftrightarrow\) no first order energy correction

iii) \(m_e + 2\omega_S = -1\): same as ii)
Finally

\[ m_e + 2m_s = 2 : \quad (e=1, m_e=1, m_s=\frac{1}{2}) \]

\[ \Sigma L \left| \begin{array}{c} 1 \, (\frac{1}{2}) \\ 1 \, (\frac{1}{2}) \end{array} \right| = (1)(\frac{1}{2}) \left| \begin{array}{c} 1 \, 1 \, (\frac{1}{2}) \end{array} \right| , \text{ the other terms vanish} \]

\[ \Rightarrow E^{(i)}_3 = \frac{1}{2} \lambda \]

\[ m_e + 2m_s = -2 : \quad \left| \begin{array}{c} 1 \, -1 \, -\frac{1}{2} \end{array} \right| \]

\[ E^{(i)}_4 = (-1)(-\frac{1}{2}) \lambda = \frac{1}{2} \lambda \quad \checkmark \]