Quantum operator methods
Adapted from Physics 3220 Tutorial
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Without specifying how measurements are to be performed, imagine that we measure the components of identical quantum particles’ spins in either the \( x \) or \( z \) direction. The hermitian operator \( \hat{S}_x \) that measures spin in the \( x \) direction has only two orthonormal eigenvectors, with eigenvalues \( \pm 1 \), labelled as

\[ \hat{S}_x |\rightarrow\rangle = |\rightarrow\rangle \quad \hat{S}_x |\leftarrow\rangle = -|\leftarrow\rangle \]  

(1) Just to make sure we’re all on the same page, what are the eigenvectors in the above equations, what are the corresponding eigenvalues, and what are \( \langle \rightarrow | \rightarrow \rangle \), \( \langle \leftarrow | \rightarrow \rangle \), \( \langle \rightarrow | \leftarrow \rangle \) and \( \langle \leftarrow | \leftarrow \rangle \)?

Now consider the hermitian operator \( \hat{S}_z \) that measures spin in the \( z \) direction. As with \( \hat{S}_x \), this operator has two orthonormal eigenvectors with eigenvalues \( \pm 1 \), but these eigenvectors differ from those of \( \hat{S}_x \). We label them

\[ \hat{S}_z |\uparrow\rangle = |\uparrow\rangle \quad \hat{S}_z |\downarrow\rangle = -|\downarrow\rangle \]  

(2) As above, what are the eigenvectors in the above equations, what are the corresponding eigenvalues, and what are \( \langle \uparrow | \uparrow \rangle \), \( \langle \downarrow | \uparrow \rangle \), \( \langle \uparrow | \downarrow \rangle \) and \( \langle \downarrow | \downarrow \rangle \)?

While the eigenvectors of \( \hat{S}_z \) differ from those of \( \hat{S}_x \), suppose they are related by

\[ |\uparrow\rangle = N (|\rightarrow\rangle + |\leftarrow\rangle) \].

(3) Calculate the numerical constant \( N \). Is your answer unique? If not, does it matter?

(4) Starting with an arbitrary state \( |\psi\rangle \), suppose a measurement of \( \hat{S}_z \) produces eigenvalue 1. What state is the particle in now? How much ambiguity is there in your answer?
(5) Measure $\hat{S}_z$ again: what results can you get, with what probabilities? What state is the particle in after this second measurement? How much ambiguity is there in your answer?

(6) Now imagine measuring $\hat{S}_x$ after those two measurements of $\hat{S}_z$. Calculate the probability you will obtain the ($\hat{S}_x$) eigenvalue -1.

(7) Completeness dictates that $|\downarrow\rangle = a |\rightarrow\rangle + b |\leftarrow\rangle$. Calculate the numerical constants $a$ and $b$. Is your answer unique? If not, does it matter? (If your results for $N$ in (3) were not unique, take $N$ to be real and positive.)

(8) Starting with an arbitrary state $|\psi\rangle$, suppose a measurement of $\hat{S}_x$ produces eigenvalue -1. What state is the particle in now? How much ambiguity is there in your answer?

(9) Calculate the probability that a subsequent measurement of $\hat{S}_z$ will yield eigenvalue -1.

(10) Given an ensemble of particles in state $|\rightarrow\rangle$, measure $\hat{S}_z$ on each of them independently. Do all these measurements give the same result? If not, what are the possible results, with what probabilities? What is the average result of all these independent measurements?
Define the matrix representation in the $\hat{S}_z$ basis by

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

\textbf{(11)} What is the matrix representation of $\hat{S}_z$ in this basis?

\textbf{(12)} What are the matrix representations of $|\rightarrow\rangle$ and $|\leftarrow\rangle$ in this $\hat{S}_z$ basis? (If your results for $a$ and $b$ in (7) were not unique, take $a$ to be real and positive.) Use these to determine the matrix representation of $\hat{S}_z$ in this $\hat{S}_z$ basis.

\textbf{(13)} Do $\hat{S}_x$ and $\hat{S}_z$ commute? Use the matrix representation you determined above to calculate $[\hat{S}_x, \hat{S}_z]$. The three hermitian $2 \times 2$ matrices $\hat{S}_x$, $\hat{S}_z$ and $\frac{i}{2} [\hat{S}_x, \hat{S}_z]$ should look familiar – who are they named after?

\textbf{(14)} Based on the above, does measuring $\hat{S}_z$ on some particle affect the outcome of a subsequent measurement of $\hat{S}_z$? Does measuring $\hat{S}_x$ on some particle affect the outcome of a subsequent measurement of $\hat{S}_z$? How does this depend on our choice of representation and basis?

\textbf{(15)} What are $\langle \uparrow \mid$, $\langle \downarrow \mid$, $\langle \rightarrow \mid$ and $\langle \leftarrow \mid$ in this matrix representation? Use this matrix representation to calculate the probability that measuring $\hat{S}_z$ on $|\rightarrow\rangle$ gives -1.
Let’s define another operator
\[ \hat{\sigma}_- |\uparrow\rangle = |\downarrow\rangle \quad \hat{\sigma}_- |\downarrow\rangle = 0 \]
(16) Are \(|\uparrow\rangle\) and \(|\downarrow\rangle\) eigenstates of \(\hat{\sigma}_-\)?

(17) What is \(\hat{\sigma}_-\) in the matrix representation defined above (11)? Is \(\hat{\sigma}_-\) hermitian? Does \(\hat{\sigma}_-\) correspond to something you can measure or observe?

(18) What is \(\hat{\sigma}^\dagger_-\) in this matrix representation? Use this to find \(\hat{\sigma}^\dagger_- |\uparrow\rangle\) and \(\hat{\sigma}^\dagger_- |\downarrow\rangle\).

Finally, suppose the hamiltonian commutes with \(\hat{S}_z\) but not with \(\hat{S}_x\).
(19) Starting with an arbitrary state \(|\psi\rangle\), does the probability that measuring \(\hat{S}_z\) will yield 1 (corresponding to state \(|\uparrow\rangle\)) depend on the amount of time you wait before measuring?

(20) Starting with an arbitrary state \(|\psi\rangle\), does the probability that measuring \(\hat{S}_x\) will yield 1 (corresponding to state \(|\rightarrow\rangle\)) depend on the amount of time you wait before measuring?