1. Here we will explore some properties of angular momentum eigenstates (10 pts:5+5):

(a) Show that for eigenstates of $\hat{L}_z$, denoted as $|l, m\rangle$, the expectation values of $\hat{L}_+, \hat{L}_-, \hat{L}_x$ and $\hat{L}_y$ vanish, and the expectation values of $\hat{L}_x^2$ and $\hat{L}_y^2$ are equal

(b) Show that these states satisfy the uncertainty relation.

\[ \Delta L_x \Delta L_y \geq \frac{|\langle [\hat{L}_x, \hat{L}_y] \rangle|}{2} \quad (1) \]

Show that when $m = \pm l$ the equality is satisfied. Comment.

2. A neutron, magnetic moment $\vec{\mu} = \mu_n \hat{S}$, travels with velocity $\vec{v} = 2400 \text{ m/s}$ along the $\hat{z}$ direction. Initially it travels in a region (region 1) of uniform magnetic field $B_0 \hat{z}$ with spin up in the $z$ direction. At time $t_1$ it enters another region (region 2) where the magnetic field is $\vec{B} = B_0 (\cos \theta \hat{z} + \sin \theta \hat{x})$. After traveling for 20 m in this region it enters a third region where the magnetic field is once again $B_0 \hat{z}$. Assuming that $\mu_n B_0 = \pi 60 \text{s}^{-1}$ answer the following questions (30 pts:10+10+10):

(a) Calculate the state, $|\psi(t_1)\rangle$, of the neutron at time $t_1$

(b) Use $|\psi(t_1)\rangle$ as an initial condition to evaluate the state of the neutron in region 2 as a function of time.

(c) Calculate the probability that the spin is down in the $z$ direction after the neutron has traveled through region 2.
3. The operator that rotates the spin of a $S = 1/2$ particle by an angle $\varphi$ about an axis of rotation along a unit vector $\mathbf{n}$ is

\[ \hat{R}(\varphi, \mathbf{n}) = e^{-i \varphi \mathbf{n} \cdot \hat{\vec{\sigma}} / 2} \]  

(2)

where $\hat{\vec{\sigma}} = \{\sigma_x, \sigma_y, \sigma_z\}$ is the Pauli matrices vector. $\hat{R}(\varphi, \mathbf{n})$ is a $2 \times 2$ matrix that transforms the two component spinor wave function upon rotation. (30 pts: 6 each)

(a) Show that

\[ \hat{R}(\varphi, \mathbf{n}) = I \cos(\varphi/2) - i(\mathbf{n} \cdot \hat{\vec{\sigma}}) \sin(\varphi/2) \]  

(3)

here $I$ is the identity matrix (Hint: Expand the exponent in a Taylor series: remember $e^{ix} = 1 + ix + (ix)^2/2! + \ldots + (ix)^k/k! + \ldots = \cos x + i \sin x$).

(b) To see that this operator really rotates the physical system consider the particular case when $\mathbf{n}_y = (0, 1, 0)$. Show that if we apply $\hat{R}(\varphi, \mathbf{n}_y)$ to an electron with spin up along $z$, the vector $\{\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle\}$ changes as expected from a classical rotation.

(c) What is $\hat{R}(\varphi = 2\pi)$ and $\hat{R}(\varphi = 4\pi)$. Comment.

(d) In an experiment the following sequence is applied to an electron initially in spin up along $\hat{z}$.

\[ |\psi(T)\rangle = \hat{R}(-\pi/2, \mathbf{n}_y) e^{-i\omega T \hat{\vec{\sigma}} \cdot \hat{\vec{\sigma}} / 2} \hat{R}(\pi/2, \mathbf{n}_y) |\uparrow\rangle_z \]  

(4)

Calculate $\langle \psi(T) | \hat{S}_z | \psi(T) \rangle$ at the end of the experiment.

(e) Use $\hat{R}(\varphi, \mathbf{n})$ to calculate the eigenstates of the operator $\mathbf{e} \cdot \hat{\vec{S}} = \hat{S}_x \sin \varphi + \hat{S}_z \cos \varphi$ which satisfy

\[ \mathbf{e} \cdot \hat{\vec{S}} |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle. \]  

(5)