Solution Homework 3

1.) According to the addition of angular momenta, \( S = S_1 + S_2 \), can take values from \( S_1 + S_2, ..., |S_1 - S_2| \).

In the case of two spin 1 particles \( S = 2, 1, 0 \)

Let's look at these states using the Clebsch-Gordan coefficients table we have in the book.

\[
\begin{align*}
\frac{5}{2}, \frac{5}{2} & = \begin{pmatrix} \pm 1, \pm 1 \end{pmatrix} \\
\frac{1}{2}, \frac{1}{2} & = \begin{pmatrix} \pm \frac{1}{2}, \pm \frac{1}{2} \end{pmatrix} \\
1, 0 & = \begin{pmatrix} \pm 1, 0 \end{pmatrix}
\end{align*}
\]

Here I will use the convention to compute:

\[
\begin{align*}
|\frac{5}{2}, \frac{5}{2}\rangle &= \frac{1}{\sqrt{10}} (|\pm 1, \pm 1\rangle + |\pm \frac{1}{2}, \pm \frac{1}{2}\rangle) \\
|\frac{1}{2}, \frac{1}{2}\rangle &= \frac{1}{\sqrt{3}} (|\pm 1, 0\rangle + |\pm 0, -1\rangle) \\
|1, 0\rangle &= \frac{1}{\sqrt{2}} (|1, 1\rangle - |1, -1\rangle)
\end{align*}
\]

It is clear from these wave functions that \( S=2 \) states are symmetric under particle exchange.

Now let's look at \( S = 1 \):

\[
\begin{align*}
|\frac{1}{2}, \frac{1}{2}\rangle &= \frac{1}{\sqrt{2}} (|\pm 1, 0\rangle + |\pm 0, 1\rangle) \\
|1, 0\rangle &= \frac{1}{\sqrt{2}} (|1, 1\rangle - |1, -1\rangle) \\
|1, -1\rangle &= \frac{1}{\sqrt{2}} (|0, 1\rangle - |1, 0\rangle)
\end{align*}
\]

These are anti-symmetric under the exchange of two particles.

\( S=0 \):

\[
|0, 0\rangle = \frac{1}{\sqrt{3}} (|1, 1\rangle - |1, -1\rangle + |0, 0\rangle + |1, 1\rangle) \quad \text{symmetric under exchange}
\]
Now if the spatial part of the wave function is symmetric, since we are dealing with bosons, the spin part must be symmetric as well. It implies that only the $S=0, 2$ values are allowed.

2) a) Assuming symmetric spatial wave function we need the spin part to be symmetric as well then

- All $m_s = 1$: $\mid \uparrow \rangle = \mid \uparrow \rangle \mid \uparrow \rangle \mid \uparrow \rangle$

- $2m_s = 1$ and $1m_s = 0$: $\mid \Psi \rangle = \frac{1}{\sqrt{3}} (\mid 0+\rangle + \mid 0+\rangle + \mid 1+\rangle)$

- Different $m_s$: the wave function can have $3 \cdot 2 \cdot 1 = 6$ terms

$$\mid \Psi \rangle = \frac{1}{\sqrt{6}} (\mid 0+\rangle + \mid -0\rangle + \mid 0-\rangle + \mid 10\rangle + \mid 1-\rangle + \mid -1\rangle)$$

b) the case $m_s = 1$ for all particles is not possible because we cannot construct an anti-symmetric wave function.

- $2m_s = 1$, and $1m_s = 0$: this is also not possible since we have two identical $m_s = 1$ states

- All different is the only possible
a simple way to construct it is using a determinant

$$\det \begin{bmatrix} + & 0 & - \\ + & 0 & - \\ + & 0 & - \end{bmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{6}} \left( |10\rangle + |0\rangle + |0\rangle + |0\rangle + |1\rangle + |1\rangle \right)$$

3) For an Infinite wall potential the stationary wave functions are

$$\Psi_n(x) = \frac{\sqrt{2}}{\sqrt{L}} \sin \left( \frac{n\pi x}{L} \right) \quad k_n = \frac{n\pi}{L}$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2L^2 m}$$

a) If the spin of the particle is a triplet the ground state requires to have the two lowest energy states populated which have $$n = 1, n=2$$

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left( |\Psi_1(x_1)\Psi_2(x_2) - \Psi_2(x_1)\Psi_1(x_2)| S=1, M \right)$$

$$E = \frac{\hbar^2 \pi^2}{2L^2 m}$$

b) If the system is in the singlet spin wave function it spatial part can be symmetric, so the lower energy state is

$$\Psi(x_1, x_2) = |\Psi_1(x_1)\Psi_1(x_2)| S=0, M=0$$

$$E = \frac{\hbar^2 \pi^2}{L^2 m}$$
The singlet state has lower energy than the triplet state.

c) Let think first about $S=1$. In this case the orbital part has to be antisymmetric

$$\Delta E = -\lambda \int \psi^\dagger(x_1, x_2) \, \mathcal{H}(x_1 - x_2) \, \psi(x_1, x_2) \, dx_1 \, dx_2$$

$$= -\lambda \int |\psi^\dagger(x_1, x_1)|^2 \, dx$$

$$\psi^\dagger(x_1, x_1) = 0 \quad \Rightarrow \quad \Delta E = 0$$

This implies that triplet particles are not affected by interactions. Energy remains the same.

This is not the case if particles are in a singlet state

$$\Delta E = -\lambda \int \left( \sqrt{\frac{2}{L}} \, \sin \left( \frac{\pi x}{L} \right) \right)^2 \, dx$$

$$= -\lambda \int \left[ \frac{2}{L} \sin^2 \left( \frac{\pi u}{L} \right) \right] \, du$$

$$= -\frac{2\lambda}{L} \int \sin^2(u) \left( 1 - \cos^2(u) \right) \, du$$

$$= -\frac{2\lambda}{L} \left[ \int_0^\pi \left( 1 - \cos(u) \right) \, du \right] - \frac{2\lambda}{L} \left[ \int_0^\pi \sin^2(u) \, du \right]$$

$$= -\frac{2\lambda}{L} \left[ \frac{\pi}{2} - \frac{\pi}{8} \right]$$

For the singlet state, the energy is reduced.

This is expected because when we have an attractive potential, the particles minimize their energy by being close together and they do that better in the singlet-state.