Physics 4410 Homework #7
Due Wednesday, Oct. 15, IN CLASS. Recall: late homework will not be accepted.
Be sure to show your work and explain what you are doing.

1) (20 points) Here’s a good example of the use of the variational principle. Consider an oscillator that is not harmonic, but quartic, with Hamiltonian

$$H = \frac{p^2}{2m} + \lambda x^4$$

This is not so much fun to solve directly, but the variational principle can get you near the ground state. Using a Gaussian trial wave function, make a variational estimate of the ground state of this oscillator. Compare your answer to the exact result, which is

$$E_0 \approx 1.060\lambda^{1/3}\left(\frac{\hbar^2}{2m}\right)^{2/3}$$

In particular, is your guess high or low?

2) (10 points) Suppose $V(x)$ is an attractive potential in one dimension, i.e., it is negative and tends to zero as $|x| \to \infty$. Use the variational principle to show that $V(x)$ always has a bound state. This is a remarkable fact: no matter how weakly negative $V(x)$ is, it always has a bound state! (Remark: this is not necessarily true in three dimensions.)

3) (10 points) Here’s a useful fact about approximate wave functions. Suppose $|\psi_0\rangle$ is the true ground state of a Hamiltonian $H$, i.e., $H |\psi_0\rangle = E_0 |\psi_0\rangle$. Suppose further that you’ve guessed a very nearly exact approximate wave function $|\psi\rangle = |\psi_0\rangle + \epsilon |\phi\rangle$, where $|\phi\rangle$ is orthogonal to $|\psi_0\rangle$ and $\epsilon$ is a small parameter.

a) How can you justify choosing $|\phi\rangle$ orthogonal to $|\psi_0\rangle$?

b) Show that the variational estimate to the ground state energy, using the trial function $|\psi\rangle$, is accurate to terms of order $\epsilon^2$. That is, energies tend to be more accurate than wave functions.