Physics 4410 Homework #3
Due Wednesday, Sept. 17, IN CLASS. Recall: late homework will not be accepted.
Be sure to show your work and explain what you are doing.

1) (10 points) Suppose the operator \( A = (\hat{S}_1 \cdot \hat{S}_2)^2 \) acts on the system of two spin-1/2 particles, \( s_1 = s_2 = 1/2 \). What are the eigenvalues and eigenstates of \( A \)?

2) (10 points) The Hamiltonian for a two-state system is given in matrix form by
\[
H = \begin{pmatrix}
E_1 & \Delta \\
\Delta & E_2
\end{pmatrix}
\]

a) Diagonalize this \( H \) to find its eigen-energies.
b) Using perturbation theory, find the eigen-energies of \( H \) through order \( \Delta^2 \).
c) Expand your solution from a) in a power series in \( \Delta \), also to order \( \Delta^2 \) and show that this result agrees with your answer from part b).

3) (20 points) When is a spring not exactly a spring? To a lowest-order approximation, you can think of the bonds between two atoms in a molecule as represented by a harmonic oscillator potential \( V(r) \approx \frac{1}{2} m\omega^2 (r-r_e)^2 \), where \( r \) stands for the distance between the atoms, and \( r_e \) is their equilibrium separation. In other words, you can pretend the atoms are connected by a tiny spring with spring constant \( k = m\omega^2 \). However, a more realistic potential for a diatomic molecule is given by the Morse potential,
\[
V(r) = D [1 - \exp(-a(r-r_e))]^2.
\]
a) Sketch the Morse potential, labeling \( D \) and \( r_e \) on your sketch. Measured from the minimum at \( r=r_e \), how much energy is required to pull the atoms apart?
b) Make a Taylor expansion of the Morse potential around its minimum. If you approximate the potential by a harmonic oscillator potential, \( V(r) \approx \frac{1}{2} m\omega^2 (r-r_e)^2 \), what is its ground state energy, as measured from the bottom of the potential? Hint: to carry out the Taylor expansion, it is handy to use the shifted coordinate \( x=r-r_e \).
c) The next order term of the Taylor expansion is proportional to \( x^3 \). Why does this term not contribute to the ground state energy, in first-order perturbation theory?
d) The next order term is proportional to \( x^4 \), and this term does contribute in first-order perturbation theory. What is the resulting energy shift of the ground state?

Remark: the Taylor expansions of the Morse potential are somewhat tedious, and are not the point here. If you want to do these using your favorite software, I won’t stop you.