1) Consider a particle confined in a time-modulated harmonic oscillator, described by

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega(t)^2 x^2 \]

Where \(\omega(t) = \omega_0 + \delta \omega \cos(ft)\) and \(\delta \omega \ll \omega_0\). At time \(t = 0\) the oscillator is in its ground state \(|0\rangle\). Using first-order perturbation theory, find the probability that the particle is in an excited state at time \(t > 0\). You may use the fact that \(\langle n | x^2 | 0 \rangle = \frac{\hbar}{\sqrt{2m\omega_0}}\) for \(n = 2\), and is zero for any other value of \(n\).

2) Consider a gas of hydrogen atoms in their ground state, including in their \(F = 0\) hyperfine ground state. The gas is initially inside a magnetic field, which is turned off at time \(t = 0\). The field in the magnetic field coils decays exponentially in time, as \(\tilde{B}(t) = \tilde{\zeta} B_0 \exp(-t/\tau)\). For this problem, you may neglect the interaction of the magnetic field with the nuclear spin.

a) Can this perturbation excite the atoms to any of the \(F = 1\) hyperfine states? If so, which ones?

b) Your answer to a) was “yes.” So, after a long time, what fraction of the atoms are in an excited hyperfine state?

3) In class we argued that, at long times, the transition probability from the ground state to the excited state was a delta function in the frequency, \(\delta(\omega_b - \omega_a - \omega)\). This is not really the case, however, and the reason is that the excited state suffers from spontaneous emission. We can model this phenomenologically as follows.

a) Let the excited state energy have an imaginary part, i.e., \(\omega_b \rightarrow \omega_b - i\gamma/2\). Show that this represents decay from the excited state, with time constant \(\tau = 1/\gamma\).

b) Now adapt our derivation of the Fermi Golden Rule to incorporate this complex energy. Show that for some time \(T\) long compared to \(1/\gamma\) you arrive at a frequency dependence

\[ P_{\alpha \rightarrow \beta} \propto \frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2} \]

This Lorentzian shape represents a more realistic line shape for photoabsorption. (Note: This quick derivation is obviously not adequate, since this probability is quickly no longer small compared to one. But it does illustrate the relation between lifetimes and linewidths.)