Problem 9.1: Uranium and polonium. (15 points)

Nuclei consist of a bound state of protons and neutrons. The number of protons is the atomic number $Z$ and determines which element we are talking about. Even for fixed $Z$ we can still vary the atomic weight $A$, which is the number of protons plus neutrons. Nuclei with the same $Z$ but different $A$ are said to be different isotopes of the same element. Isotopes are generally indistinguishable as far as atomic physics goes, but can be very different when it comes to nuclear properties such as radioactivity.

a) The isotope $^{238}\text{U}$ (where $A = 238$ is the atomic weight) is radioactive and decays via $\alpha$-decay. Indicate what isotope of what nucleus it decays into, and use the formulas coming from Gamow’s theory of $\alpha$-decay to obtain a numerical estimate of its lifetime.

The size of the nucleus is on the order of a Fermi ($1 \text{ fm} = 10^{-15} \text{ m}$) and the volume scales with the atomic weight; you may use $r_1 \simeq (1.07 \text{ fm})A^{1/3}$. Estimate the kinetic-plus-potential energy $E$ of the $\alpha$-particle as the difference between the mass-energy of $^{238}\text{U}$, using $E = mc^2$, and the sum of the mass energies of the $\alpha$-particle and the other product nucleus, which you should look up (and keep enough significant digits!). For the velocity, just take $E = m_\alpha v^2/2$; this ignores the potential energy inside the nucleus but is good enough for our purposes.

b) Do the same for $^{212}\text{Po}$.

c) How do your results compare to the experimental lifetimes of $\tau = 6 \times 10^9 \text{ years}$ and $\tau = 0.5 \mu\text{s}$, respectively? Which isotope do you exact to be present in greater abundance on Earth?

Problem 9.2: Quantized energies for general potential wells. (15 points)

a) Consider a generalized infinite square well, where the “floor” of the well need not be flat, but instead is some general $V(x)$ in between $x = 0$ and $x = a$; but let $V = \infty$ for $x < 0$ and $x > a$ as usual. Write down a general classically allowed WKB wavefunction for the region $0 < x < a$, and use the fact that the infinite walls require $\psi(x = 0) = \psi(x = a) = 0$ to show that the energy $E$ must satisfy the quantization condition,

$$\int_0^a p(x)dx = n\pi\hbar,$$  \hspace{1cm} (1)

for positive integer $n$, where $p(x) = \sqrt{2m(E - V(x))}$ as usual. Why is $n \leq 0$ not possible?
b) For the infinite square well, the infinite walls happen suddenly, and so the WKB approximation is valid right up to them. For a general potential well this is not the case. A careful treatment of the matching formulas between classically allowed and classically forbidden regions modifies the quantization condition you found in the previous part to

\[ \int_{x_-}^{x_+} p(x) dx = \pi \hbar \left( n - \frac{1}{2} \right), \]  

(2)

where \( x_\pm \) are the classical turning points and \( n \) is still a positive integer. For an example, let’s consider a system we already know the answer to — the simple harmonic oscillator. Recall that in this case \( V(x) = (1/2) m \omega^2 x^2 \). For a state with energy \( E \), what are the classical turning points in terms of \( E \) and other constants? Use the updated WKB formula (2) for the quantization of energies to estimate the energies of the harmonic oscillator. Compare them to the exact answer; what do you need to use for the minimum value of \( n \)?

c) For which values of \( n \) (large, small, medium, something else?) do you expect the WKB approximation to give the best results? Comment on how this addresses the discrepancy between the quantization formulas (1) and (2).

Problem 9.3: Probability current and magnetic field. (15 points)

For the Hamiltonian of a particle without a magnetic field (so \( \vec{A} = 0 \)) we can define a probability current \( \vec{j} \) as:

\[ \vec{j}(\vec{r},t) \equiv -\frac{i\hbar}{2m} \left( \Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^* \right) = \frac{\hbar}{m} \text{Im} \left( \Psi^* \vec{\nabla} \Psi \right), \]

(3)

which represents the flux of probability flowing past a point: probability per unit time per unit area.

a) Consider the case where there is a magnetic field present, so \( \vec{A} \neq 0 \). The Schrödinger equation is

\[ \frac{1}{2m} \left( -i \hbar \vec{\nabla} - q \vec{A}(\vec{r}) \right)^2 \Psi(\vec{r},t) + V(\vec{r}) \Psi(\vec{r},t) = i\hbar \frac{\partial}{\partial t} \Psi(\vec{r},t). \]

(4)

Recall that the values of \( \vec{E} = -\vec{\nabla} \varphi - \partial \vec{A} / \partial t \) and \( \vec{B} = \vec{\nabla} \times \vec{A} \) as well as the form of the Schrödinger equation are invariant under gauge transformations,

\[ \varphi' = \varphi - \frac{\partial \Lambda}{\partial t}, \quad \vec{A}' = \vec{A} + \vec{\nabla} \Lambda, \quad \Psi' = e^{i q \Lambda / \hbar} \Psi, \]

(5)

for any function \( \Lambda(\vec{r},t) \).

Write down how \( \vec{j}' \) is related to \( \vec{j} \), and so demonstrate that \( \vec{j} \) is not gauge invariant.
b) The results of part a) indicate that $\vec{j}$ is no longer a well-defined physical quantity when $\vec{A} \neq 0$, since it depends on the gauge chosen. Find a term $\Delta \vec{j}$ to add to $\vec{j}$ such that the total,

$$\vec{j}_A \equiv \vec{j} + \Delta \vec{j}, \quad (6)$$

is gauge-invariant.

c) When $\vec{A} = 0$, the probability current and the probability density $\rho(\vec{r},t) \equiv |\Psi(\vec{x},t)|^2$ (which is gauge-invariant already) satisfy the continuity equation, encoding that probability density only changes when the probability current flows in or out. Show that when $\vec{A} \neq 0$ it is $\vec{j}_A$ that satisfies the continuity equation with $\rho$,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j}_A = 0, \quad (7)$$

confirming that it is indeed $\vec{j}_A$ that is the correct probability current in the more general case. *Hint:* use the Schrödinger equation (including the $\vec{A}$ parts) to eliminate terms like $\partial \Psi / \partial t$. 
