Problem 10.1: Hamiltonian with vector potential. (15 points)

a) Show that the Lorentz force law,
\[ \vec{F} = q\vec{E} + q\vec{v} \times \vec{B}, \]  
(1)
can be written as
\[ m\ddot{x}_a = -q\frac{\partial \phi}{\partial x_a} + q\vec{v} \cdot \frac{\partial \vec{A}}{\partial x_a} - q\frac{dA_a}{dt}, \]  
(2)
where \( a = x, y, z \) is one of the components of the vector. A helpful vector identity is
\[ [\vec{v} \times (\vec{\nabla} \times \vec{A})]_a = \vec{v} \cdot \frac{\partial \vec{A}}{\partial x_a} - (\vec{v} \cdot \vec{\nabla})A_a. \]  
(3)
You will also need to think carefully about the difference between the partial derivative \( \partial / \partial t \) and the total derivative \( d/dt \).

b) The Hamiltonian for a charged particle in a vector potential \( \vec{A} \) and a scalar potential \( \phi \) is
\[ H = \frac{1}{2m} (\vec{p} - q\vec{A})^2 + q\phi. \]  
(4)
Show that Hamilton’s first set of equations,
\[ \frac{dx_a}{dt} = \frac{\partial H}{\partial p_a}, \]  
(5)
reproduce the relationship
\[ \vec{p} = m\vec{v} + q\vec{A}. \]  
(6)
Notice in the presence of the vector potential the “momentum” used by the Hamiltonian formalism, called the canonical momentum, is not just the mass times the velocity (called the kinetic or kinematic momentum). It is the canonical momentum that is conjugate to position in the sense of the Hamiltonian formalism, and in the quantum theory, the canonical momentum which has the standard commutation relation with position \([x_a, p_b] = i\hbar \delta_{ab}\).

c) Using the result of part b), show that Hamilton’s other set of equations,
\[ \frac{dp_a}{dt} = -\frac{\partial H}{\partial x_a}, \]  
(7)
reproduce the Lorentz force law as you expressed it in part a). Recall that in the Hamiltonian formalism, it is \( p_a \) that is kept constant when one differentiates with respect to \( x_a \), and vice versa.

**Problem 10.2:** A magnetic monopole. (20 points)

Some useful formulas in spherical coordinates:

\[
\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi},
\]

\[
\nabla \times \vec{X} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \sin \theta X_\phi - \frac{\partial X_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial X_r}{\partial \varphi} - \frac{\partial}{\partial r} (r X_\varphi) \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r X_\theta) - \frac{\partial X_r}{\partial \theta} \right] \hat{\phi}.
\]

Consider the vector potential

\[
\vec{A} = \frac{g(1 - \cos \theta)}{r \sin \theta} \hat{\phi}.
\]

a) First, show that the associated magnetic field \( \vec{B} \) is that for a magnetic monopole, that is, a Coulomb magnetic field. What quantity plays the role of “magnetic charge”?

b) If there is a magnetic Coulomb field, then the divergence of the magnetic field is not zero at the origin. However, we expect that any divergence of a curl of something well-defined is always zero. We should therefore ask what went wrong.

What went wrong is that the vector potential is not well-defined everywhere. Where besides the origin is the vector potential \( \vec{A} \) singular? Be sure to treat naively indeterminate quantities carefully. This locus of singularities is called the Dirac string.

c) If we have a magnetic monopole we can never get rid of the Dirac string, but gauge transformations can move it around. Define the new vector potential

\[
\vec{A}' = \frac{g(-1 - \cos \theta)}{r \sin \theta} \hat{\phi}.
\]

Find a gauge transformation relating \( \vec{A}' \) to \( \vec{A} \). Where is the Dirac string for \( \vec{A}' \)?

Other gauge transformations can move the string to other locations. Since the location of the Dirac string is arbitrary and it only shows up in \( \vec{A} \), never in \( \vec{B} \), it is not a physical object.

d) Imagine a particle with ordinary electric charge of magnitude \( q \), moving around in the presence of the magnetic monopole; this electrically charged particle has wavefunction \( \Psi \). When using the two different vector potentials \( \vec{A}' \) and \( \vec{A} \) we must use different wavefunctions, \( \Psi' = h \Psi \) for a function \( h \). What is \( h \) in this case?
Assuming the wavefunction $\Psi$ is well-defined, the gauge transform $\Psi'$ will only be well defined if $h(\varphi = 0) = h(\varphi = 2\pi)$, since $\varphi = 0, 2\pi$ represent the same point. What constraint must we place on $q$ and $g$ to ensure $\Psi'$ is well-defined? This constraint is called the Dirac quantization condition. If a magnetic monopole exists (none has ever been observed, but a wide class of so-called grand unified theories predicts they might), what must we conclude about every electric charge in the universe?

**Problem 10.3**: Landau levels. (20 points)

Consider a uniform magnetic field pointing in the $z$-direction, $\vec{B} = B\hat{z}$, and a particle of charge $q$ moving in the $x$-$y$ plane in the presence of this field. The particle is constrained not to move in the $z$-direction, so we may drop all $z$-derivatives.

a) To write down the Schrödinger equation we need to choose a vector potential $\vec{A}$. (Since $\vec{E} = 0$, we will always choose the scalar potential $\varphi = 0$.) One choice for the uniform magnetic field that we have already used in class is $\vec{A}' = (1/2)\vec{B} \times \vec{r} = (B/2)(x\hat{y} - y\hat{x})$. However, we will find a different choice of gauge to be more useful: consider the gauge choice $\vec{A} = Bx\hat{y}$. Find the gauge transformation parameter $\Lambda$ relating this choice of gauge to $\vec{A}'$, thereby proving that they lead to the same $\vec{B}$.

b) Consider the time-independent Schrödinger equation for a charged particle moving in the $x$-$y$ plane in the presence of the vector potential $\vec{A} = Bx\hat{y}$, and write a trial stationary state in the variable-separated form,

$$\psi(x, y) = f(x)g(y). \tag{10}$$

Which of $p_x$ and $p_y$ commutes with the Hamiltonian? Choose $\psi$ to be an eigenvector of the commuting momentum, and thus determine $g(y)$ (it will depend on a single quantum number) — don't bother normalizing $g(y)$. Substitute the resulting form of $\psi$ back into the TISE to find an equation for $f(x)$.

c) Verify that with a constant shift in the $x$-coordinate,

$$x' \equiv x - x_0, \tag{11}$$

where $x_0$ is a constant you need to determine in terms of other quantities, the TISE equation from the previous part can be cast in the form of a simple one-dimensional Schrödinger equation with a familiar potential. Indicate what kind of potential it is, and what $x_0$ needs to be for this to work.

d) What are the resulting energy eigenvalues of the full system, and what are their degeneracies? These states are called Landau levels.

e) The degeneracy found above can be reduced by confining the particle to a finite region on the $x$-$y$ plane. Say it is confined to a square defined by $0 \leq x \leq X$ and $0 \leq y \leq Y$, ...
with $X$ and $Y$ constant lengths. Impose periodic boundary conditions in the $y$-direction and thus produce a constraint on the quantum number characterizing $g(y)$. Now assume that the shift of the $x$-coordinate must live inside the available region,

$$0 \leq x_0 \leq X,$$

(12)

and use this to derive another constraint on the same quantum number characterizing $g(y)$, and explain how this reduces the degeneracy. (Thus if you have sufficiently many charged fermions in a magnetic field in a finite region, the lowest Landau level eventually gets filled up and higher levels become occupied.)

(Side note: Landau levels underlie the quantum physics of electrons moving in a magnetic field, and in particular are at the root of the famous quantum Hall effect.)

**Problem 10.4:** Harmonic oscillator in transient electric field. (15 points)

Consider a one-dimensional harmonic oscillator with frequency parameter $\omega_0$. At time $t = 0$, the system is in the ground state. Then at time $t = 0$, a constant electric field of magnitude $E_0$ is turned on, and left on until it is turned off at time $t = T$. The oscillator responds to the electric field because it has charge $q$:

$$H'(t) = -q E_0 x, \quad 0 \leq t \leq T,$$

$$= 0 \quad \text{other times.} \quad \text{(13)}$$

At later times, the system will be in the state

$$\Psi(x, t) = \sum_{n=0}^{\infty} c_n(t) e^{-iE_n t/\hbar} \psi_n(x), \quad \text{(15)}$$

where $\psi_n(x)$ and $E_n$ are as usual the eigenvectors and eigenvalues of the SHO Hamiltonian.

a) Calculate the matrix elements $H'_{n0}$ between the state $\psi_n$ and the ground state $\psi_0$. (As usual for the SHO, raising and lowering operators make things simplest.) Which $H'_{n0}$ are nonzero?

b) Calculate the coefficients $c_n(t)$ to first order using time-dependent perturbation theory, and find the probabilities at first order to transition to each state $\psi_n$ after time $t = T$. (Hint: before you calculate, think about which ones are nonzero.)

c) At which order would the ability to transition to $\psi_2$ first appear?