1. **Spin ½ perturbation theory.** (25 points)

The spin ½ systems have been a focus in the class because they have about the simplest Hilbert Space you can imagine, yet they show the full range of behavior that we’ve come to expect from quantum systems, like non-commuting operators, etc. Why not see how things behave when a small term is added to a spin ½ Hamiltonian?!

You should now be pretty comfortable with the case of a spin ½ in a magnetic field, with an applied field in the $z$-direction and of magnitude, $B_0$. The Hamiltonian for that case is:

$$\hat{H}^0 = -\gamma B_0 S_z$$

Notice that I’ve labeled the Hamiltonian to indicate that it is some ‘zero-order’ Hamiltonian that we know the behavior of. Often people say, “We know the solution.” What they mean, of course, is that they know the eigen state vectors and the associated eigen energies for the Hamiltonian. SO:

a) **List the zeroth-order solutions:** What are the eigen vectors and eigen energies for this Hamiltonian?

b) Now we are ready to consider the changes that happen due to a small perturbation to the zeroth-order problem. Let’s consider a situation where we apply, in addition to the magnetic field in the $z$-direction, a small component of magnetic field in the $x$-direction. The Hamiltonian is now more general. We can still write it in vector form as:

$$\hat{H} = -\gamma \vec{B} \cdot \vec{S}$$

Let’s assume that:

$$\vec{B} = B_0 \hat{z} + B_x \hat{x}$$

Write down the perturbation contribution to the Hamiltonian, $H’$, assuming that the field component in the $x$-direction is small.

c) Calculate the first-order energy shift of the two energy levels.

d) Calculate the first-order corrections to the energy eigen state vectors. Are the new vectors still orthogonal?

e) Calculate the second-order corrections to the two energy levels.
2. Two coupled harmonic oscillators. (25 pts.)

Simple harmonic oscillation is a great approximation to many things, say like the vibrations of a mass on a spring, or the vibrations of atoms in a molecule, or vibrations of the electromagnetic field, and so on. Many things behave like a nearly harmonic oscillator. Also, harmonic oscillators can ‘talk to each other’. The oscillations of one harmonic oscillator (say excitations of a microwave electromagnetic field) can couple to another harmonic oscillator (say the vibrations of water molecules in your cold coffee) and Hey! Now you’re cooking…

OK, so let’s consider a problem where we have two mass-spring harmonic oscillators that are coupled.

The basic Hamiltonian is:

$$\hat{H}^0 = \frac{\hat{p}_1^2}{2m_1} + \frac{1}{2} k_1 \hat{x}_1^2 + \frac{\hat{p}_2^2}{2m_2} + \frac{1}{2} k_2 \hat{x}_2^2$$

Let’s assume that the masses and spring constants are equal. Since the oscillators have the same spring constant and the same mass, they also have the same frequency of oscillation.

a) Write down a representation for the energy eigen state vectors and eigen energies for this unperturbed Hamiltonian. Don’t use the Gaussian/Hermite spatial representation. Instead, use the ‘number representation’ kets from the operator algebra treatment of the form $|n_1, n_2\rangle$. Write down in this representation the ground state vector, ground state energy, and the first excited state vectors and energies. Which are degenerate and which are non-degenerate (if either).

b) Next, let’s couple the oscillators by adding a term to the Hamiltonian that looks like:

$$\hat{H}' = \frac{1}{2} \gamma (\hat{x}_1 - \hat{x}_2)^2$$

If the factor, $\gamma$, is sufficiently small, we can try using perturbation theory to understand what this coupled system does.

Look at the Hamiltonian and explain physically what this perturbation piece represents. How might you build a system with this type of extra energy? It might be useful to ask what the units of $\gamma$ might be.

c) Use non-degenerate perturbation theory to calculate the first order shift of the ground state energy.
d) Use non-degenerate perturbation theory to calculate the second order shift of the ground state energy.

e) Use degenerate perturbation theory to calculate the first order shifts of the first excited state energies. What are the two energies and what are two associated eigen state vectors (again only to first order). Make a plot of the shifts of the first excited state energies versus $\gamma$. 