In each problem, be sure to give the reasoning for your answer and define any variables you create. If you use a general formula, state that formula clearly before manipulating it.

1. [5 pts] Two systems, A and B, are placed in thermal contact with each other, but are otherwise isolated. Initially $T_A > T_B$. Make no assumptions about the dependence of $S$ on $U$ for either system. Determine whether each statement is true or false, including your reasoning. For example, you might cite a relevant thermodynamic law, or give a simple counterexample.
   A. A and B will always eventually come to the same temperature $T$.
   B. Heat will always flow from system A to system B.
   C. The entropy of each system will always increase.
   D. The energy of the total system will be minimized.
   E. The entropy of the total system will be maximized.

2. [3 pts] Two experiments are performed. In the first, an ideal gas of $N$ atoms at temperature $T_i$ in volume $V_i$ is allowed to expand to a final volume $V_f$ in an adiabatic, quasi-static process. In the second experiment, which has the same initial conditions, the gas undergoes free expansion to the same final volume. Which experiment caused the greatest change in entropy? Why?

3. [3 pts] Consider a system with entropy $S$ which is a function of $U$, $V$, and $N$. Derive a formula for the chemical potential for the case of constant energy and entropy.

4. [3 pts] The dependence of entropy $S$ on energy $U$ for each of two identical systems A and B is shown in the figure below; the state of each system is indicated by the labeled dot. Which system has the higher temperature? Give your reasoning.

\[ S \quad A \quad B \]

\[ U \]
5. [8 pts] 100 g of ice at -200°C is placed in a thermally insulated pool with 1,000,000,000 g of water at 25°C. Find the total entropy change of the ice plus pool system after it reaches thermal equilibrium. You may assume all processes are at constant volume. Data: $c_{\text{ice}} = 2.1 \text{ J/(g°C)}$, $c_{\text{water}} = 4.2 \text{ J/(g°C)}$, $l_{\text{ice-water}} = 335 \text{ J/g}$.

6. In a paramagnet with spin-1 dipoles, each dipole can take on 3 states: $-1$, $0$, and $+1$, relative to the direction of the magnetic field, with energies $-\mu B$, $0$, and $+\mu B$, respectively. Consider a spin-1 dipole in thermal equilibrium with a reservoir at temperature $T$.

   (a) [5 pts] Derive an expression for the partition function for this dipole.

   (b) [4 pts] Give expressions for the probabilities of the dipole being in each state.

   (c) [4 pts] Determine and interpret the low and high temperature limits for your answers to part (b).
7. The Maxwell speed distribution is derived using a classical, non-relativistic expression for the kinetic energy. If we follow the canonical method and introduce relativity, we should instead use an expression for the total relativistic energy for a free particle, \( E = \sqrt{p^2 + m^2} \), where I have taken the speed of light \( c = 1 \) for algebraic simplicity. The (3D vector) \( \mathbf{p} \) of magnitude \( p \) is the particle momentum and \( m \) is the rest mass.

(a) [5 pts] Derive an (integral) expression for the partition function \( Z \) for a relativistic ideal gas atom in three dimensions at temperature \( T \). DO not attempt to evaluate the integrals! You can assume that \( kT \) is large enough to ensure that the atom is in the “classical regime” in the statistical mechanics sense.

(b) [5 pts] Explaining your reasoning, find an expression for the probability distribution \( D(p) \, dp \) for the relativistic ideal gas at temperature \( T \). This distribution gives the probability of finding a gas particle with momentum of magnitude \( p \) between \( p \) and \( p + dp \). (Hint: Recall that \( d^3p = p^2 \, dp \sin \theta \, d\theta \, d\phi \). You do not need to insert any explicit expression for \( Z \) from part (a) in your answer.

(c) [3 pts] Rather than evaluate the exact expression from part (b), we study the deviations from the non-relativistic case by expanding the square root for the case \( p \ll m \). (Remember that with the \( c \)'s included we are really looking at the case \( pc \ll mc^2 \), so don’t worry that \( p \ll m \) is not much of a limit). Expand the square root keeping terms up to \( p^4 \) and modify your answer from part (b) to use this approximation. Again, do not attempt to evaluate any integrals. (Recall \( (1 + x)^n \approx 1 + nx + n(n - 1)x^2/2 + ... \) for \( x \ll 1 \).) Interpret your answer to part (b). In particular, what is the effect of the addition of a large constant term (the rest mass energy) to the regular non-relativistic case? Does it affect the probability distribution in \( p \) at a given \( T \)? What will be the effect of the higher order \( p \) term on \( D(p) \, dp \), compared to the non-relativistic case?

(d) [2 pts] Now consider the opposite limit of very large \( p, \ p \gg m \). What will be the functional behavior of \( D(p) \, dp \) in this limit (to first order only)? Compare this to the non-relativistic case as well.