Measurement of the hydrogen spectrum with a reflection grating spectrometer

Purpose
The aim of the experiment is to measure the wavelengths of visible lines in the hydrogen spectrum using a reflection grating spectrometer. The spectrometer uses a converging beam configuration. The results will be used to test the validity of the Blamer formula and to determine the Rydberg constant and the Ionization Energy for hydrogen. We will aim for between 1% and 0.1% accuracy.

References

Introduction
1) The Hydrogen Atom
Quantum mechanics predicts that the hydrogen atom possesses energy levels given by the formula:

\[ E_n = -\left( \frac{me^4}{2(4\pi\varepsilon_0\hbar)^2} \right) \frac{1}{n^2} = -\frac{E_i}{n^2} \]

Where \( E_i \) is the ionization energy. The principal quantum number, \( n \), can take only the integer values 1, 2, 3, \( \cdots \infty \). The observed spectral lines comprise photons emitted in downward quantum jumps, whose frequencies, \( \nu \), are given by the Planck relation. \( \Delta E = h\nu \).

The frequency of photons emitted in the quantum jump from atomic state, \( i \), to state, \( f \), is:

![Energy levels and spectral series in the hydrogen atom.](image.png)
\[ v_{i,f} = \frac{E_f}{\hbar} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \]

Since the photon wavelength and frequency are related by \( \lambda = \frac{c}{\nu} \), we can write:

\[ \frac{1}{\lambda_{i,f}} = \frac{\Delta E}{\hbar c} = R_e \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \]

where

\[ R_e \equiv \left( \frac{m e^4}{2(4\pi \varepsilon_0)^2 \hbar^2 c} \right) \]

Here, \( R_e \) is the Rydberg constant for the case of an infinitely massive nucleus. For a real hydrogen atom, \( m \) needs to be replaced by the reduced mass of the electron, \( m' \).

\[ m \to m' = \frac{m m_p}{m + m_p} \]

where \( m_p \) is the proton mass.

The Rydberg constant for atomic hydrogen is:

\[ R_H = \frac{m_p}{m + m_p} R_e \]

A spectral series contains all possible transitions to the same final state.

The Lyman series, terminating in the ground state \( (n_F=1) \) all have wavelengths in the ultraviolet. The visible lines of the hydrogen spectrum all terminate in the \( n=2 \) state, and are known as the Balmer series. The Paschen series \( (n_F=3) \) lies in the infrared region of the spectrum.

2) Spectrum Formation with a transmission grating

In this experiment, you will be measuring the photon wavelengths emitted from a hydrogen plasma. To measure these wavelengths, you will be using a reflection grating spectrometer. In your introductory physics classes, you have learned about transmission gratings and we begin our discussion of grating spectrometers with this simpler transmission case.

When parallel light falls normally on a transmission grating whose spatial period is \( 'a' \), a principle maxima of diffracted intensity occur at those angles for which the path difference for light through adjacent slits is an integral number of wavelengths. The diffraction angle, \( \theta_n \), for a path difference of \( n \) wavelengths is given by:
\[ a \sin \theta_n = n\lambda \]

\( n \) is the order of the principal maximum. Separation of colors occurs because the angle is a function of wavelength. The widths, \( \Delta \theta \), of principal maxima are given by, \( \Delta \theta \approx \lambda/W \cos \theta \), for a total grating with, \( W \).

The light to be analyzed in a spectrometer is first focused on an entrance slit, then converted to a parallel beam by the collimator lens. All light diffracted at the same angle, \( \theta_1 \), by the grating possesses the same wavelength, \( \lambda_1 \). A focusing lens focuses all light parallel to \( \theta_1 \) to a single point, \( P_1 \), in its focal plane. Light of another wavelength, \( \lambda_2 \), is diffracted through a different angle, \( \theta_2 \), and will be focused at another point, \( P_2 \). The points \( P_1 \) and \( P_2 \) are separated in the image plane by distance, \( \Delta x = (\theta_2 - \theta_1) f_T \).

This style of spectrometer suffers from inefficient use of the input light: In real transmission gratings, much of the incident beam is simply reflected, rather than appearing in the transmitted direction. Further, of the light transmitted, much appears in the central maximum (\( n=0 \)) for which there is no separation of colors (no dispersion). The small amount that is dispersed becomes shared among different orders of spectra. Our weak hydrogen lamps can be seen only in zeroth order, and dispersed spectra are not visible with a transmission grating.

3) The blazed reflection grating

Reflection gratings solve most of the practical problems listed above for transmission gratings. The blazed grating permits a significant fraction of all the light to be diffracted into a single order.

For light reflected by any grating, principal maxima occur at angles for which the path difference between adjacent rays is equal to an integer number of wavelengths, \( n\lambda \). For plane waves incident at an angle, \( \theta_i \), and reflected at an angle, \( \theta_r \), a principal maximum of reflected light occurs if the angles satisfy the grating equation:

\[ a(\sin \theta_{r,n} - \sin \theta_i) = n\lambda \]

As is shown in Figure 7.3, \( \theta_r \) and \( \theta_i \) are defined as...
positive when they are on opposite sides of the normal to the grating. $\theta_{r,n}$ is the angle at which the reflected wave creates the $n$th order maximum. Note that for zeroth order ($n=0$), we have: 
$$a\left(\sin \theta_{r,0} - \sin \theta_{r,0}\right) = 0$$
or $\theta_{r,0} = \theta_{r,0}$. Then, the angle of incidence is equal to the angle of reflection; the zeroth order reflection is mirror-like.

Figure 7.3, above, emphasizes the Huygens principal of constructive interference of spherical waves, by noticing point-like scattering centers on a regular spacing. However, real reflection gratings have some particular surface shape that is reproduced periodically. By choosing the particular surface shape, we can have a significant impact on the amount of light diffracted into the different diffraction orders. Blazed gratings have a saw-toothed periodic pattern characterized by a particular angle and depth. The grating is optimized for a particular wavelength and is designed to be operated in the ‘back reflection geometry’, where $\theta_i = -\theta_{r,n}$ i.e., the diffracted reflected wave leaves with an angle to the grating normal that is equal to the incident angle. The diffracted wave travels directly back toward the light source. The grating is also designed to achieve efficient diffraction of the chosen wavelength into this particular direction. To achieve this high efficiency, the grating is designed as shown in Figure 7.4: The saw-tooth pattern is such that the surface normal of each step points in the direction of the preferred diffraction angle and the depth of each step is $\lambda/2$ to cause constructive interference between wave scattered from different steps. Gratings of this design can diffract over 80% of the incident power into the preferred direction (relative to the amount reflected by a simple mirror made from the same materials). Typical blazed reflection gratings are cut into plastic or glass surfaces and are then aluminized to increase reflection. A typical grating might be optimized for a wavelength in the center of the visible spectrum, say for $\lambda = 500$ nm.

As a brief comment, we note that the design of blazed reflection gratings is a procedure that suffers from the so-called 'inversion problem': Given a known surface profile, it is a straightforward process to calculate the far-field electromagnetic intensity pattern via Maxwell's equations. However, given a desired far-field intensity pattern, there is no simple process to follow in designing the required surface.

**Apparatus**

A diagram of the major components of the apparatus are shown below in Figure 7.5
Figure 7.5. Layout for the complete “Converging beam reflection grating spectrometer”. P₁, P₂, and P₃ are the fixed points of the systems.

<table>
<thead>
<tr>
<th>Components</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input system:</td>
<td>S Source (gas discharge spectral tube with H or Hg)</td>
</tr>
<tr>
<td>LC Condenser Lens (projector lens FL=100 mm, f/3.5)</td>
<td></td>
</tr>
<tr>
<td>S' Image of S</td>
<td></td>
</tr>
<tr>
<td>Spectrometer:</td>
<td>T Slit (razor blades)</td>
</tr>
<tr>
<td>E Black screen (with white stip on back for spectrum display)</td>
<td></td>
</tr>
<tr>
<td>T' Images of T (for the different wavelengths!)</td>
<td></td>
</tr>
<tr>
<td>D Photodiode detector (vertically above T)</td>
<td></td>
</tr>
<tr>
<td>LS Spectrometer lens (Zeiss 210 mm, f 5.6 camera lens)</td>
<td></td>
</tr>
<tr>
<td>E' White screen (removable)</td>
<td></td>
</tr>
<tr>
<td>T'' Image of T (with grating removed)</td>
<td></td>
</tr>
<tr>
<td>Optical lever:</td>
<td>He-Ne Laser</td>
</tr>
<tr>
<td>M Flat mirror (attached with wax to back of grating)</td>
<td></td>
</tr>
<tr>
<td>R Meter rule (attached to wall with double stick tape)</td>
<td></td>
</tr>
</tbody>
</table>
Principles of operation

1. Input system

Light from source, S, is a spectral tube in which atoms of a low pressure gas (P roughly 0.1 to 1 torr) are excited by electron collisions in an electric discharge. The bright region of the discharge is about 1 mm diameter by 10 cm high. The condenser lens, LC, images S onto the entrance slit, T, of the spectrometer in such a way as to maximize the amount of light transmitted by the spectrometer.

2. Spectrometer

Slit T, acts as a line source of light for the spectrometer. In the absence of a grating, the spectrometer lens, LS, produces an image, T", of the slit on screen, E'. In Fourier optics, we have seen that a transmission grating placed after the lens at any distance, d, from the screen, gives rise to a Fraunhofer diffraction pattern in the plane of the screen, E'. For monochromatic light, there is a single line for each order of diffraction \( n = \cdots -1, 0, +1, +2, \cdots \) whose position depends on wavelength, \( \lambda \). For light containing a mixture of wavelengths, each diffraction order is split into as many lines as there are wavelengths. In this experiment, the transmission grating is replaced by a blazed reflection grating, C, to increase the intensity of diffracted light. If G is placed at distance, d, from diffraction plane E', then the reflected diffraction pattern, T', is formed at an equal distance, d, to the left of G. For convenience, we place the reflected spectral plane in the same plane as slit, T.

Photodetector, D, is placed in the spectral plane vertically above slit, T, to measure the wavelength of a particular line, the grating, G, is rotated about the vertical axis until the photodiode registers a maximum of intensity. The grating equation gives:

\[
a \left( \sin \theta_{r,n} - \sin \theta_i \right) = n \lambda
\]

For our geometry (as shown in Figure 7.6), with D directly above T, we have \( \theta_i = -\theta_{r,n} \), so the value of wavelength is determined from the equation:

\[
2a \sin \theta_i = n \lambda
\]

3. Optical lever

The angle of the grating can be accurately determined by reflecting a laser beam off a mirror, M, attached to the grating onto a meter rule, perpendicular to OA. Then,

\[
\tan (2\theta_i) = \frac{y}{z}
\]

This commonly used device is known as an 'optical lever'.

Figure 7.6. Top view of the geometry for the blazed diffraction grating.
Outline of the Experiment

1. **Set up the spectrometer (using the mercury lamp and laser)**
   - Define the optic axis by the laser beam centered on the spectral lamp.
   - Establish the fixed points P1 (source), P2 (plane of slit and spectral display), and P3 (plane of image of slit).
   - Adjust the condenser lens, LC, to image S onto T. Adjust the spectrometer lens to image T onto E'.
   - Install grating G and adjust to give sharp image of zero order line on screen E'.

2. **Calibrate the grating and validate the theory of the grating using the mercury lamp.**
   - Use the zero order line to set the zero for the optical lever.
   - Measure the angles, \( \theta_i \), for all visible mercury lines for orders \( n = \pm 1, \pm 2 \).
   - Graph \( \sin \theta_i \) vs. \( n\lambda \) and determine the grating period, \( a \), to +/- 0.25%. Is the theory correct?

3. **Measure the wavelengths for the hydrogen spectrum.**
   - After changing the lamp, adjust position to procure an image on T.
   - Check zero for the zero order spectrum.
   - Measure the angles, \( \theta_i \), for all visible lines plus the near UV lines if detectable. Repeat these measurements for \( n = \pm 1, \pm 2 \) if possible.
   - Calculate wavelengths at the bench and repeat any doubtful measurements.

4. **From your data:**
   - Test the Rydberg formula - finding \( n_F \) from the value that gives straight lines in the graphs of \( 1/\lambda_i \) vs. \( (1/n_F^2 - 1/n_i^2) \) for various choices of \( n_F \).
   - Find the best choice of \( R_H \) from the slope and compare with expectations.
   - Find the photon energies for each line.
   - Find the ionization energy for hydrogen, \( E_i \).
Problems

1. **Design of spectrometer.**
   Assume collimator lens has 100 mm focal length and an f-number (f/D) of 3.5, that the spectrometer lens has a 210 mm focal length and f-number of 5.6, and that the grating has 1000 lines/mm and a grating size of 25 mm by 25 mm.

   a) Calculate the distances between the fixed points (l1 and l2 in Figure 5.6) for magnification $|m1| = 1/3$ for LC and $|m2| = 3$ for LS. What are the approximate positions for the lenses? Calculate distance, d, for the grating when the image, T', is formed in the same plane as the input slit, T.

   b) Calculate the diffraction limited resolving power $\lambda/\Delta\lambda$ for the given grating (use the first order diffraction peak).

   c) Calculate the value of slit width that gives a 1% resolving power for geometrical resolution.

   d) Calculate the angles, $\theta$, for 400 nm and 700 nm light, for diffraction orders $n = 1, 2, 3$. Draw a map of the lines and explain how you would resolve ambiguities caused by overlapping orders.

2. **Light intensity.**
   The hydrogen lamp is relatively weak. At a distance of 60 cm the intensity is roughly 2 $\mu$W/cm$^2$. It appears to fall off as $1/r^2$ at this distance. The discharge is 1 mm diameter by 10 cm in height.

   a) Calculate the total power emitted by the lamp in mW.

   b) Estimate the power falling on the photodiode when the slit width has the value calculated in part 1 (c).
Useful Formulas for grating problems

Grating equation: \[ a \left( \sin \theta_{r,a} - \sin \theta_i \right) = n \lambda \]

or
\[ 2a \sin \theta_i = n \lambda \] for back reflection.

Diffraction limited resolving power: \[ \frac{\lambda}{\Delta \lambda} = \frac{nW_g}{a} \]
where \( W_g \) is the grating width.

Angular dispersion: \[ \frac{d \theta_r}{d \lambda} = 2 \tan \theta_r / \lambda \text{ (rad/nm)} \]
where \( \theta_r = \theta_i \)

Linear dispersion: \[ \frac{d \lambda}{d \theta} = 2d \tan \theta / \lambda \text{ (rad/nm)} \]

Geometrical resolution: \[ \frac{\Delta \lambda}{\lambda} = (W_{r'} / \lambda) \left( \frac{\Delta \lambda}{\Delta x} \right) = (m2W_s / \lambda) \left( \frac{\Delta \lambda}{\Delta x} \right) \]
where \( W_{r'} \) is width of image of slit on plane E'.
\( W_s \) is width of slit.

Throughput of spectrometer: (Slit area)(solid angle into lens) or \[ T = A_S \Omega_{LS} = \left( w_s h_s \right) \left( \pi D^2 / 4 f_i^2 \right) \]

Radiance of source: Power per unit area per unit solid angle \[ L_s = \frac{P_s}{4\pi (10\text{cm})(1\text{mm})} \]

Power reaching spectral plane: Radiance of source multiplied by throughput of the spectrometer. \[ P = L_s T \]
Data
Mercury wavelengths recommended as secondary standards

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<th>COLOR</th>
<th>Wavelength (nm)</th>
<th>Intensity (spark) arb. Units</th>
</tr>
</thead>
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<td></td>
</tr>
<tr>
<td>IR</td>
<td>1014.0</td>
<td></td>
</tr>
<tr>
<td>Yellow</td>
<td>579.054</td>
<td>1000</td>
</tr>
<tr>
<td>Yellow</td>
<td>576.959</td>
<td>200</td>
</tr>
<tr>
<td>Green</td>
<td>546.074</td>
<td>2000</td>
</tr>
<tr>
<td>Blue-Green</td>
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<td>50</td>
</tr>
<tr>
<td>Blue</td>
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<td>500</td>
</tr>
<tr>
<td>Violet</td>
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<td>150</td>
</tr>
<tr>
<td>Violet</td>
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