$$\nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E} + \nabla (\nabla \cdot \vec{E}) = -\nabla^2 \vec{E}$$
$$= -\frac{1}{c^2} \nabla \times \frac{\partial \vec{E}}{\partial t} = -\frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \vec{E}) = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \nabla \times \vec{E}$$

Hence we have \( \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \) or

\[ \nabla^2 \vec{E} - \frac{\epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \Rightarrow \nabla^2 \vec{E} - \frac{\mu}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \]

is the equation for a wave.

Similarly, we get

\[ \nabla^2 \vec{B} - \frac{\epsilon}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \]

We define the index of refraction \( n \) by \( v = \frac{c}{n} \), the velocity of light in the dielectric medium. Hence we make the connection

\[ \frac{1}{v^2} = \frac{n^2}{c^2} = \frac{\epsilon}{c^2} \quad \text{or} \quad \frac{1}{v^2} = \frac{n^2}{c^2} = \frac{\epsilon}{c^2} \]

How about the relative magnitude of \( \vec{B} \) and \( \vec{E} \). In free space (using our units), the \( \vec{B} \) and \( \vec{E} \) associated with radiation are equal in magnitude. Now in a dielectric medium

\[ \nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t} = \frac{\epsilon}{c} \frac{\partial \vec{E}}{\partial t} \]

Now the solution is \( \vec{E} = E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \)
where \( \lambda f = v = \gamma n = \gamma \sqrt{\varepsilon} \). So in the dielectric medium either \( \lambda \) or \( f \) or both need to change so that \( (\lambda f)_{\text{new}} = \frac{1}{n} \frac{c}{c} = \frac{1}{n} (\lambda f) \) in free space.

It turns out from experimental analysis and the requirement of boundary conditions at all times that \( \omega \) does not change and \( \lambda \to \gamma n \). Hence

\[
E = E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\
B = B_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad \quad \quad \quad \quad k = \frac{2\pi}{\lambda} \quad \quad \lambda = \frac{c}{f} \quad \text{free space}
\]

Hence from Maxwell equations

\[
\nabla \times B = \frac{1}{c} \frac{\partial D}{\partial t} = \frac{\varepsilon}{c} \frac{\partial E}{\partial t} \\
\nabla \times B = i\mu_0 n 2\pi f B = \varepsilon i 2\pi f E \\
\lambda \quad \text{c} \\
\text{or} \quad n|\vec{B}| = \varepsilon |\vec{E}| \quad \text{some} \quad \lambda f = c \quad \text{or} \quad |\vec{B}| = \sqrt{\varepsilon |\vec{E}|} = n|\vec{E}|
\]

Now we can write the solution to the wave equation for a medium with index of refraction \( n \) and traveling in the \( z \) direction is given by \( i(nkz - \omega t) \)

\[
E = E_0 e^{i(nkz - \omega t)} \quad B = B_0 e^{i(nkz - \omega t)}
\]

where \( k = \frac{2\pi}{\lambda} \) and \( \lambda, \omega \) are the wavelengths and \( \omega \)'s in free space.
We need to get a function describing the index of refraction of a dielectric material. We will only deal with uniform, isotropic dielectric materials where \( \mathbf{P}, \mathbf{E}, \) and \( \mathbf{D} \) all point in the same direction. To obtain \( n \) we get \( \varepsilon \) which can be obtained from the equations we got last semester

\[
\mathbf{D} = \varepsilon \mathbf{E} = \mathbf{E} + 4\pi \mathbf{P} \\
\text{or } \varepsilon = 1 + 4\pi \frac{\mathbf{P}}{\mathbf{E}}
\]

\[n = \left( 1 + 4\pi \frac{\mathbf{P}}{\mathbf{E}} \right)^{1/2}\]

We recall that \( \mathbf{P} \), the polarization, is due to the displacement of the electrons from their equilibrium position due to the incident \( \mathbf{E} \) field. We consider a

"Simple Picture"

Suppose there are \( N \) atoms/cm\(^3\) each with one effective electron which can be displaced from its equilibrium position by the incident \( \mathbf{E} \) field producing a dipole moment. Now the electron is effectively being attracted back to the atom by the nucleus which has a slightly more positive charge than the remaining electrons. This means that the electrons are bound to the atom.

\[V = \frac{1}{2} k x^2\]

The electron is at the bottom of the potential well.
For small displacements the electron executes simple harmonic motion which is described by a potential $V = \frac{1}{2}kx^2$.

Hence, assuming the electron experiences a restoring force $-bx$, the equation of motion becomes

$$F = -eE = -m\frac{d^2x}{dt^2}$$

$\text{SHM motion}$

It turns out we also have something like a frictional force (like Lenz law) which damps the oscillations. The source of this damping is the radiation emitted by the electron executing the SHM motion. Hence the full equation of motion of the electron in the presence of an incident electric field $E$ in the $x$ direction is

$$m\frac{d^2x}{dt^2} = -eE - bx - ma\frac{dx}{dt}$$

$\alpha = $ damping coeff.

The constant $b$ is usually written in terms of the free electron oscillation

$$m\frac{d^2x}{dt^2} = -bx \Rightarrow \omega_0^2 = \frac{b}{m}$$

$\omega_0 = $ natural electron oscillation frequency

Now we note that in the case of forced oscillations the electric field has a time dependence $E = e^{-i(kx - \omega t)}$ and hence $x$ and $F$ also have a time dependence $e^{i\omega t}$.
Therefore the electron has
\[ \frac{dx}{dt} = i \omega x \]

Therefore \[ \frac{d^2 x}{dt^2} = -\frac{eE}{m} - \omega_0^2 x - a \frac{dx}{dt} \]
\[ -\omega^2 x = -\frac{eE}{m} - \omega_0^2 x - i \omega ax \]

Now what is the \( E \) that the electron feels, namely \( E = E_{\text{effective}} \) is felt by the electron inside the material; namely the electron feels the incident electric field + the polarization field of all the other electrons (atoms) in the material or dielectric.

We calculated this last semester

\[ E_{\text{effective}} = E + 4\pi \mathbf{P}/3 \]

where \( \mathbf{P} \) is the polarization per unit volume = \(- N_e x \) (since the electron charge is negative). Hence we get

\[ -\omega^2 x = -\frac{e E_{\text{effective}}}{m} - \omega_0^2 x - i \omega ax \]
\[ = -\frac{e}{m} \left( E - 4\pi N_e x \right) - \omega_0^2 x - i \omega ax \]

or

\[ x = \frac{eE/m}{\omega^2 - \omega_0^2 + 4\pi N_e^2 - i \omega m} \]
\[ \kappa(t) = x e^{i \omega t} = \frac{e^{i \omega t}}{m} \frac{E e^{i \omega t}}{\omega^2 + 4\pi N e^2 - \omega_0^2 - i\alpha \omega} \]

Now we can write for the dielectric constant:

\[ \varepsilon = 1 + 4\pi \frac{P}{E} = 1 + \frac{4\pi N e^2}{E} \]

\[ = 1 + \frac{4\pi N e^2/m}{\omega_0^2 - 4\pi N e^2 - \omega^2 + i\alpha \omega} \]

\[ = \sqrt{\varepsilon} = \left\{ 1 + \frac{4\pi N e^2/m}{\omega_0^2 - \omega^2 + i\alpha \omega} \right\}^{\frac{1}{2}} = 1 + \frac{2\pi N e^2/m}{\sqrt{\omega_0^2 - \omega^2 + i\alpha \omega}} \]

The index of refraction is a function of the frequency or wavelength of the radiation (light of 3600 < \lambda < 8000 Angstroms (1 Angstrom = 10^{-8} cm)). There is resonance when \( \omega = \omega_0 \), the natural frequency of the dielectric \( \omega_0 \).

The graph shows the real part of the index of refraction, \( n_{\text{real}} \), as a function of frequency, \( \omega \), with \( \omega_0 \) being the frequency of resonance. Most dielectrics are here, visible frequency.
When \( \nu < \omega \), \( n_{\text{real}} > 1 \), and when \( \nu > \omega \), \( n_{\text{real}} < 1 \). Most dielectrics we deal with in optics have \( n_{\text{real}} > 1 \). Note that in the region where \( n \) varies rapidly we have the case where slightly different wavelengths travel at different velocities \( V = \nu n \). We are in the region of anomalous dispersion.

What happens in the region where \( n < 1 \) and \( \nu n > c \). We have to distinguish between the phase velocity of light and the group velocity of light. The theory of relativity deals with the group velocity, and this \( \nu < c \) even when we are in the region when \( n < 1 \). So the phase velocity can be \( > c \) even when the group velocity \( \nu < c \).

Behavior of Radiation at the Interface Between 2 Dielectric Surfaces

Let us consider the case that one side of the dielectric interface has index of refraction \( n = n_1 \), and the other side has \( n = n_2 \). The incident radiation has an electric and magnetic field perpendicular to the vector of propagation and \( \vec{E} \times \vec{B} \) is in the direction of propagation. We consider only the case where the material is non-magnetic. \( D = \varepsilon E = n_2 E \) and \( H = B \).

\[
E = E_0 e^{i(kz - \omega t)}
\]

where \( k = \frac{2\pi}{\lambda} \), \( \lambda \) = wavelength in vacuum and \( \vec{z} \) points in the direction of propagation.

\[
\frac{\omega}{c} = \frac{2\pi}{2\pi f} = \frac{1}{2f} = \frac{1}{v_{\text{phase}}}
\]
Consider a wave approaching an interface. The most general case is that some of it is reflected and some of it is transmitted. We make no assumptions about the reflected and transmitted wave.

We do not assume that any combination of the \( \vec{k} \) vectors form a plane.

In principle, the incident, reflected and transmitted radiation are a priori unpolarized but any direction of the \( \vec{E} \) fields from \( \vec{k} \) can be resolved into two vectors (components) on two directions to each other. So we will consider only the case where the \( \vec{E} \) field is in one of two perpendicular directions.

We write the general case

\[
\begin{align*}
\vec{E}_i &= \vec{E}_{0i} e^{i(\vec{n}_i \cdot \vec{k}_i - \omega t)} \\
\vec{E}_r &= \vec{E}_{0r} e^{i(\vec{n}_r \cdot \vec{k}_r - \omega t)} \\
\vec{E}_t &= \vec{E}_{0t} e^{i(\vec{n}_t \cdot \vec{k}_t - \omega t)}
\end{align*}
\]

So far we do not know \( |\vec{k}_i| = |\vec{k}_r| = |\vec{k}_t| \) or \( \vec{k}_i, \vec{k}_r, \vec{k}_t \) form a plane, \( \omega_i = \omega_r = \omega_t \), \( \vec{E}_{0i} = \vec{E}_{0r} = \vec{E}_{0t} \). All we know are the boundary conditions of the \( E, D, B, H \) fields at the uncharged surface between two dielectric surfaces.
1 component of $D$, $B$ is continuous.

Tangential component of $H$ and $E$ is continuous.

$H = B$ in non-magnetic materials.

Hence, for example, the vector sum of the tangential components of the incident and reflected $E$ fields must be equal to the transmitted tangential component $E$ and so on.

These equalities in the boundary conditions must hold for all points along the boundary and for all times. This can only be satisfied if the vectors $E_i$, $E_r$, $E_t$, $B_i$, $B_r$, $B_t$ are identical functions of $\mathbf{r}$ and $t$, otherwise the boundary conditions can not be satisfied at all locations on the interface and at all times. Hence we conclude

1) The statement that the $E$ and $B$ vectors have the same time dependence means that

$$W_i = W_p = W_t$$

2) The statement that the $E$ and $B$ vectors have the same space dependence at the interface means

$$N_1 \mathbf{E_i} \cdot \hat{\mathbf{r}} = N_1 \mathbf{E_r} \cdot \hat{\mathbf{r}} = N_2 \mathbf{E_t} \cdot \hat{\mathbf{r}}$$

We choose our origin of coordinates to be on the interface, and choose the $z$ direction to be the normal to the plane defining the interface. Then the $x, y$ plane is the plane of the interface.
So \( \mathbf{p} \) on the surface of the interface is written

\[
\mathbf{r} = x\mathbf{i} + y\mathbf{j} \\
k_i = -k_i \cos \Theta_i \mathbf{i} + k_i \sin \Theta_i \mathbf{k}
\]

where we define the \( \mathbf{i} \mathbf{k} \) plane to include the direction of the incident radiation.

The requirement from the boundary conditions (3) demands that

\( \mathbf{k}_i, \mathbf{k}_p, \) and \( \mathbf{k}_t \) form a plane that includes the \( z \) direction. Otherwise you can not obey the boundary conditions everywhere on the surface of the interface, Plane of Incidence.

The requirement

\[
\mathbf{k}_p \cdot \mathbf{r} = \mathbf{k}_i \cdot \mathbf{r}
\]

implies

\[
x\mathbf{k}_i \cdot \sin \Theta_i = x\mathbf{k}_p \cdot \sin \Theta_p \quad \text{since} \quad \gamma \text{ cancels}
\]

Since \( |\mathbf{k}_i| = |\mathbf{k}_p| = \frac{\lambda}{n} \)

\[\sin \Theta_i = \frac{\sin \Theta_p}{n} \quad \Theta_i = \Theta_p \quad \text{Law of Reflection}
\]

The requirement

\[
x\mathbf{k}_i \cdot \mathbf{n}, \sin \Theta_i = x\mathbf{k}_t \cdot \mathbf{n} \sin \Theta_t
\]

Since \( |\mathbf{k}_i| = |\mathbf{k}_t| = \frac{\lambda}{n} \) \( \lambda \) in vacuum

\[\gamma \sin \Theta_i = n_2 \sin \Theta_2
\]

This is Snell's Law or Law of Refraction.

Hence the Law of Reflection and Refraction are a consequence of obeying the electromagnetic boundary conditions everywhere on the surfaces of the interface.
The Fresnel Equations

These equations discuss the fraction of the incident energy flux ($S = \text{Poynting Vector}$) that is reflected and the fraction that is transmitted. We want to do this as a function of the incident, reflected and refracted, or transmitted angle. Recall that $B = nE$ in a dielectric medium.

We consider two cases for the direction of the $E$ field.

1) $E$ is perpendicular to the plane of incidence

![Diagram showing vector $E_i$, $B_i$, $E_r$, $B_r$, $E_t$, and $B_t$.]

Assume this direction. If the solution has a -sign, then it means direction is opposite. $\times$ means $B$ into page. Recall $E \times B$ points along $k$.

2) $E$ vector is in the plane of incidence

![Diagram showing vector $E_i$, $B_i$, $E_r$, $B_r$, $E_t$, and $B_t$.]

Any other direction of $E$ is the vector sum of the radiation of these two cases.
Case 1: \( \mathbf{E} \) perpendicular to the plane of incidence

The boundary condition \( \mathbf{E}_{\text{tangential}} \) is continuous implies

\[
E_i + E_r = E_t
\]

they are in the same direction

hence scalar form applies

Since the time and space dependence has to be the same

for all three vectors

\[
E_{oi} + E_{or} = E_{ot}
\]

The boundary condition perpendicular component of \( \mathbf{B} \) is

continuous following the same argument of the time and space

dependence

\[
B_{oi} \sin \theta_i + B_{or} \sin \theta_r = B_{ot} \sin \theta_o
\]

or

\[
n_i E_{oi} \sin \theta_i + n_i E_{or} \sin \theta_r = n_o E_{ot} \sin \theta_o
\]

Tangential component of \( \mathbf{H} = \mathbf{B} \) is continuous

\[
B_{oi} \cos \theta_i - B_{or} \cos \theta_r = B_{ot} \cos \theta_o
\]

\[
n_i E_{oi} \cos \theta_i - n_i E_{or} \cos \theta_r = n_o E_{ot} \cos \theta_o
\]

Since \( \theta_r = \theta_i \) and \( n_i \sin \theta_i = n_2 \sin \theta_o \) equation 2 becomes

\[
E_{oi} + E_{or} = E_{ot}
\]

So equations 1 and 2 are the same.
Equation 3 becomes

\[ \frac{E_{oi} - E_{op}}{E_{it}} = \frac{\eta_2 \cos \Theta_i}{\eta_1 \cos \Theta_i} \]

where \( \eta_2 \sin \Theta_t = \eta_1 \sin \Theta_i \)
\( \eta_2 / \eta_1 = \sin \Theta_i / \sin \Theta_t \)

Using equations 1 and 3. Adding them,

\[ 2E_{oi} = 1 + \frac{\tan \Theta_i}{\tan \Theta_t} E_{ot} \]
\[ = \frac{\tan \Theta_t + \tan \Theta_i}{\tan \Theta_t} E_{ot} \]

\[ \frac{E_{ot}}{E_{oi}} = \frac{2 \tan \Theta_t}{\tan \Theta_t + \tan \Theta_i} \frac{2 \sin \Theta_t \cos \Theta_i}{\sin \Theta_i \cos \Theta_t + \sin \Theta_t \cos \Theta_i} \]

Put this solution into 3

\[ E_{oi} - E_{op} = \tan \Theta_t \left( \frac{2 \sin \Theta_t \cos \Theta_i}{\sin \Theta_i \cos \Theta_t + \sin \Theta_t \cos \Theta_i} \right) E_{oi} \]

\[ E_{oi} - E_{op} = \frac{2 \sin \Theta_i \cos \Theta_t}{\sin \Theta_i \cos \Theta_t + \sin \Theta_t \cos \Theta_i} E_{oi} \]

\[ E_{op} = \frac{\sin \Theta_t \cos \Theta_i - \sin \Theta_i \cos \Theta_t}{\sin \Theta_i \cos \Theta_t + \sin \Theta_t \cos \Theta_i} \]
4) Can also be written

\[ \frac{E_{ot}}{E_{oi}} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_t + \theta_i)} \]

5) Can also be written

\[ \frac{E_{or}}{E_{oi}} = \frac{\sin(\theta_r - \theta_i)}{\sin(\theta_t + \theta_i)} \]

Note that when \( \theta_t = \theta_i \) namely \( n_2 = n_1 \) there is no reflection namely \( \sin(\theta_r - \theta_i) = 0 \). Since \( \theta_t < \theta_i \) when \( n_2 > n_1 \) \( E_{or} < 0 \) which means \( E_{or} \) is in the wrong direction.

Using the boundary conditions for the B field we get

\[ B_{oi} \sin \theta_i + B_{or} \sin \theta_r = B_{ot} \sin \theta_t \quad \text{perp. comp continuous} \]
\[ B_{oi} \cos \theta_i - B_{or} \cos \theta_r = B_{ot} \cos \theta_t \quad \text{tang. comp continuous} \]

Using \( \theta_i = \theta_r \)

\[ B_{oi} + B_{or} = B_{ot} \frac{\sin \theta_t}{\sin \theta_i} \]
\[ B_{oi} - B_{or} = B_{ot} \frac{\cos \theta_t}{\cos \theta_i} \]

This leads

6) \[ B_{ot} = \frac{2 \sin \theta_i \cos \theta_i}{\cos \theta_i \sin \theta_t + \cos \theta_t \sin \theta_i} \cdot B_{oi} \]
Similarly

\[ B_{ox} = \cos \theta_e \sin \theta_i - \cos \theta_i \sin \theta_e \quad B_{oi} = \sin (\theta_e - \theta_i) \quad B_{oi} = \sin (\theta_e + \theta_i) \]

Here we notice also that \( B_{ox} = 0 \) when \( \theta_e = \theta_i \) or \( n_s = n_i \), namely there is no change in the mediums between interfaces.

Recalling that \( B = n \mathbf{E}_t \) from page 177 we get that eq. 6 also gives

\[ n_2 E_{ot} = \frac{2 \sin \theta_i \cos \theta_i}{\cos \theta_i \sin \theta_e + \cos \theta_e \sin \theta_i} \]

using the law of refraction \( n_1 \sin \theta_i = n_2 \sin \theta_e \) we get

\[ E_{ot} = \frac{2 \sin \theta_i \cos \theta_i}{\cos \theta_i \sin \theta_e + \cos \theta_e \sin \theta_i} \]

which is eq. 4 so all is consistent.
We now consider the 2nd case; \( E \) in the plane of incidence.

The boundary conditions are that the perpendicular component of \( D \) are continuous and tangential component of \( E \) are continuous; the perpendicular component of \( B \) is continuous and the tangential component of \( H = B \) is continuous. We recall that \( D = E = n^2 E \). Hence we have

\[
\begin{align*}
8 & \quad n_1^2 (E_{oi} \sin \theta_i + E_{or} \sin \theta_r) = n_2^2 E_{ot} \sin \theta_t \\
9 & \quad E_{oi} \cos \theta_i - E_{or} \cos \theta_r = E_{ot} \cos \theta_t \\
10 & \quad B_{oi} + B_{or} = B_{ot} \text{ or } n_1 E_{oi} + n_1 E_{or} = n_2 E_{ot}
\end{align*}
\]

which is the same as 8 after we apply the law of reflection and refraction.
Equations 8 and 9 are

\[ E_{oi} + E_{or} = \frac{n_i^2 \sin \theta_i}{n_e^2 \sin \theta_e} \quad E_{ot} = \frac{\sin \theta_i}{\sin \theta_e} E_{ot} \]

or

\[ E_{oi} - E_{or} = \frac{\cos \theta_i}{\cos \theta_e} E_{ot} \]

\[ 2E_{oi} = \left( \frac{\sin \theta_i + \cos \theta_e}{\sin \theta_e \cos \theta_i} \right) E_{ot} \]

\[ 2E_{oi} = \frac{\cos \theta_i \sin \theta_i + \cos \theta_e \sin \theta_e}{\cos \theta_i \sin \theta_i} \quad E_{ot} \]

or the fraction transmitted is

\[
\frac{|E_{ot}|}{|E_{oi}|} = 2 \left( \frac{\cos \theta_i \sin \theta_i + \cos \theta_e \sin \theta_e}{\cos \theta_i \sin \theta_i} \right) \\
\frac{|E_{ot}|}{|E_{oi}|} = \frac{2 \cos \theta_i \sin \theta_i}{\cos \theta_i \sin \theta_i + \cos \theta_e \sin \theta_e} \\
\frac{|E_{ot}|}{|E_{oi}|} = \frac{2 \cos \theta_i \sin \theta_i}{\cos \theta_i \sin \theta_i + \cos \theta_e \sin \theta_e} \\
\frac{|E_{ot}|}{|E_{oi}|} = \frac{\sin \theta_i \cos \theta_i - \sin \theta_e \cos \theta_e}{\cos \theta_i \sin \theta_i + \cos \theta_e \sin \theta_e} \\
\]

The fraction reflected is

\[ 2E_{or} = \left( \frac{\sin \theta_i - \cos \theta_e}{\sin \theta_e \cos \theta_i} \right) E_{ot} \]

leading to

\[
\frac{|E_{or}|}{|E_{oi}|} = \frac{\sin \theta_i \cos \theta_i - \sin \theta_e \cos \theta_e}{\cos \theta_i \sin \theta_i + \cos \theta_e \sin \theta_e} \\
\frac{|E_{or}|}{|E_{oi}|} = \frac{\sin \theta_i \cos \theta_i - \sin \theta_e \cos \theta_e}{\cos \theta_i \sin \theta_i + \cos \theta_e \sin \theta_e} \\
\]
Using the relation

\[
\tan(\Theta_i \pm \Theta_t) = \frac{\tan \Theta_i \pm \tan \Theta_t}{1 \pm \tan \Theta_i \tan \Theta_t}
\]

\[
\tan(\Theta_i - \Theta_t) = \frac{\tan \Theta_i - \tan \Theta_t}{1 + \tan \Theta_i \tan \Theta_t} \quad \frac{1 - \tan \Theta_i \tan \Theta_t}{\tan \Theta_i + \tan \Theta_t}
\]

\[
\tan(\Theta_i + \Theta_t) = \frac{\sin \Theta_i \cos \Theta_t - \sin \Theta_t \cos \Theta_i}{\cos \Theta_i \cos \Theta_t + \sin \Theta_i \sin \Theta_t} \times \frac{\cos \Theta_i \cos \Theta_t}{\cos \Theta_i \cos \Theta_t + \sin \Theta_i \sin \Theta_t}
\]

\[
\times \frac{\cos \Theta_i \cos \Theta_t - \sin \Theta_i \sin \Theta_t \cos \Theta_i \sin \Theta_t}{\sin \Theta_i \cos \Theta_t + \sin \Theta_t \cos \Theta_i}
\]

\[
= \frac{(\sin \Theta_i \cos \Theta_t - \sin \Theta_t \cos \Theta_i)(\cos \Theta_i \cos \Theta_t - \sin \Theta_i \sin \Theta_t)}{\cos \Theta_i \cos \Theta_t + \sin \Theta_i \sin \Theta_t}(\sin \Theta_i \cos \Theta_t + \sin \Theta_t \cos \Theta_i)
\]

\[
= \frac{\sin \Theta_i \cos \Theta_t (\cos^2 \Theta_t - \sin^2 \Theta_t) - \sin \Theta_t \cos \Theta_t (\cos^2 \Theta_i + \sin^2 \Theta_i)}{\cos \Theta_i \sin \Theta_t (\cos^2 \Theta_t + \sin^2 \Theta_t) + \cos \Theta_t \sin \Theta_t (\cos^2 \Theta_i + \sin^2 \Theta_i)}
\]

\[
= \frac{\sin \Theta_i \cos \Theta_t - \sin \Theta_t \cos \Theta_i}{\cos \Theta_i \sin \Theta_i + \cos \Theta_t \sin \Theta_t}
\]

Hence we have \( \frac{1}{E_{\Omega i}} = \frac{\tan(\Theta_i - \Theta_t)}{\tan(\Theta_i + \Theta_t)} \)
Hence we note that when \( \theta_i + \theta_e = 90^\circ \) or \( \tan(\theta_i + \theta_e) = \infty \) then \( E_p \) in the plane of incidence = 0. That means that the radiation (light) reflected only has \( E \perp \) to the plane of incidence or the light (radiation) is 100% polarized.

This angle is called "Brewster's angle." This has been tested experimentally.

\[
\frac{n_2}{n_1} = \frac{\sin \theta_i}{\sin \theta_e} = \frac{\sin \theta_i}{\sin(180^\circ - \theta_i - \theta_e)} = \frac{\sin \theta_i}{\sin(90^\circ - \theta_e)}
\]

This the case where the \( E \) field is produced an acceleration in the medium \( (n_2) \) that is \( \perp \) to the refracted ray and you are looking in the direction of the acceleration vector and there is supposed to radiation in that direction. The electron in the \( n_2 \) medium is accelerating in the direction of observation of the reflected radiation and hence no radiation is observed.
Conservation of Energy Flow at the Interface

On page 130 we showed that the Poynting vector shows the energy flow across a surface

\[ \vec{S} = \frac{1}{4\pi} (\vec{E} \times \vec{B}) \]

\[ \int_{S_i} \hat{n} \cdot d\vec{A} \] is the energy flow across the surface \( A \).

We now show that the energy associated with the radiation incident on a surface is equal to the sum of the energy being reflected + the energy being transmitted.

Total energy incident = \( S_i (A_i) = S_i \cdot w \cdot z \cdot \cos \theta_i \)
' reflected = \( S_r (A_r) = S_r \cdot w \cdot z \cdot \cos \theta_r \)
'transmitted = \( S_t (A_t) = S_t \cdot w \cdot z \cdot \cos \theta_t \)

\[ \frac{S_t \cos \theta_t + S_r \cos \theta_r}{S_i \cos \theta_i} = \text{Ratio of energy out/energy in} \]

\[ = 1 \text{ means energy out = energy in} \]
Since E and B are \( \perp \) to each other \( S = \frac{1}{4\pi} (E \times B) = \frac{1}{4\pi} E \cdot B \). The factor \( \frac{1}{4\pi} \) drops out. We solve this case for \( E \perp \) to the plane of incidence.

\[
\begin{align*}
E_t B_t \cos \theta_t + E_r B_r \cos \theta_i \quad & \quad E_i B_i \cos \theta_i \\
\end{align*}
\]

From pages 188, 189, 190

\[
= \left( 2 \sin \theta_b \cos \theta_i \sin \theta_t \cos \theta_i \cos \theta_t \right) \cos \theta_i + \left( \cos \theta_i \sin \theta_b \cos \theta_i \cos \theta_t \right)^2 \cos \theta_i \\
\left( \cos \theta_i \sin \theta_t + \cos \theta_t \sin \theta_i \right)^2 \\
= 4 \sin \theta_b \cos \theta_i \sin \theta_t \cos \theta_i \cos \theta_t + \left( \cos \theta_i \sin \theta_b \cos \theta_i \cos \theta_t \right)^2 \cos \theta_i \\
\left( \cos \theta_i \sin \theta_t + \cos \theta_t \sin \theta_i \right)^2 \\
= \left( \frac{\cos \theta_i \sin \theta_t + \sin \theta_i \cos \theta_t}{\sin \theta_t + \cos \theta_t \sin \theta_i} \right)^2 = 1 \\
\left( \frac{\cos \theta_i \sin \theta_t + \sin \theta_i \cos \theta_t}{\sin \theta_t + \cos \theta_t \sin \theta_i} \right)^2 \\
\]

The same applies when \( E \) is \( \perp \) to the plane of incidence. Note that when you are at Brewster's angle there is no reflected light. It is interesting to examine the solutions for the energy balance.
Total Internal Reflection

Snell's Law states \( n_1 \sin \theta_i = n_2 \sin \theta_e \)

What happens when \( \frac{n_1}{n_2} \sin \theta_i > 1 \) which can occur when \( n_1 > n_2 \). Then there is no real solution for \( \sin \theta_e \).

Then it turns out that all our equations for the amount of transmitted and reflected radiation are correct as long as we replace \( \theta_e \) by the relation \( \sin \theta_e \to \frac{n_1}{n_2} \sin \theta_i \).

Then

\[
\cos \theta_e \left\{ 1 - \sin^2 \theta_e \right\}^{1/2} = \left\{ 1 - \left( \frac{n_1}{n_2} \sin \theta_i \right)^2 \right\}^{1/2}
\]

\[
= i \left\{ \left( \frac{n_1}{n_2} \sin \theta_i \right)^2 - 1 \right\}^{1/2}
\]

Let us consider the case where the \( E \) field is \( \perp \) to the plane of incidence

\[
\begin{align*}
\frac{E_{o+}}{E_{o-}} &= \frac{2 \sin \theta_e \cos \theta_i}{\sin \theta_i \cos \theta_e + \sin \theta_e \cos \theta_i} \\
&= \frac{2 n_1/n_2 \sin \theta_i \cos \theta_i}{n_1/n_2 \cos \theta_i + i \left\{ \left( n_1/n_2 \sin \theta_i \right)^2 - 1 \right\}^{1/2}}
\end{align*}
\]

\[
\begin{align*}
\frac{E_{o-}}{E_{o+}} &= \frac{\sin \theta_e \cos \theta_i - \sin \theta_i \cos \theta_e}{\sin \theta_i \cos \theta_e + \sin \theta_e \cos \theta_i} \\
&= \frac{n_1/n_2 \sin \theta_i \cos \theta_i - i \sin \theta_i \left\{ \left( n_1/n_2 \sin \theta_i \right)^2 - 1 \right\}^{1/2}}{n_1/n_2 \sin \theta_i \cos \theta_i + i \sin \theta_i \left\{ \left( n_1/n_2 \sin \theta_i \right)^2 - 1 \right\}^{1/2}}
\end{align*}
\]
Note that $|E_o||E_{o1}| = 1$ and $E_{oT} = 0$. Somehow it reflects not conservation of energy flow. Pointing vector.
Similar results are obtained if $E$ is in the plane of incidence.

These equations reflect the fact that the reflected field suffers a phase change

$$e^{-2i\delta} \tan \delta = \left\{ \frac{\left( \frac{n_1}{n_2} \sin \theta_i \right)^2 - 1}{\frac{n_1}{n_2} \cos \theta_i} \right\}^{1/2}$$

The same happens when $E$ is in the plane of incidence.

We now explain the strange observation that $E_{oT} = 0$.

The solution we have for $E_{oT}$ is only at the surface of the crystal, interface. We would like to discuss what happens when we ask how the $E$ propagates away from the surface. The wave equation is

$$\nabla^2 E - \frac{n^2}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

The time dependence is $e^{i\omega t}$.

$$\nabla^2 E + \frac{n^2 \omega^2}{c^2} E = 0$$

Consider a wave propagating in the $X Y$ plane. The $\vec{E}$ part of the solution is
\[ e^{i n k \cdot r} = e^{i \left( k \sin \theta_1 x + k \cos \theta_1 y \right)} \]
\[ e^{i n_2 k \cdot r} = e^{i n_2 \left( k \sin \theta_2 x + k \cos \theta_2 y \right)} \]
\[ = e^{i \left[ i n_2 \left( n_1/n_2 \sin \theta_1 \right)^2 \cdot 1 \right]^{1/2} y} e^{i \left( n_1 \sin \theta_1 \right)} \]
\[ = e^{-n_2 \frac{2 \pi}{\lambda} \left[ (n_1/n_2 \sin \theta_1)^2 - 1 \right]^{1/2} y} e^{i \left( n_1 \sin \theta_1 \right)} \]

So it is an exponentially decaying amplitude as you propagate away from the surface.

So if \( y = 2 - 3 \lambda \), \[ e^{-n_2 \frac{2 \pi}{\lambda} \left[ (n_1/n_2 \sin \theta_1)^2 - 1 \right]^{1/2} y} \ll\ll \]

so the amplitude of \( \vec{E} \) becomes very small. So the field near the surface is not zero but it exponentially decays very quickly within a few \( \lambda \).