or \( i(n+1) \alpha = - \left\{ \frac{(Z_0 - \frac{1}{2}Z_1 - Z_2) i(n \alpha)}{Z_2} \right\} \)

Then every loop from the 0\(\text{th}\) to the \(n\text{th}\) loop obeys an equation of the form for the \(n\text{th}\) loop.

\[-Z_2 i(n-1) \alpha + (Z_1 + 2Z_2) i(n \alpha) - Z_2 i(n+1) \alpha = 0\]

This is a very symmetrical and simple expression that recurs all the time. This is known as a recursion relation. We find a general expression for \(i(n \alpha)\)

\[i(n \alpha) = A e^{n \tau} + B e^{-n \tau}\]

where \(A, B, \tau\) are complex expressions to be determined. \(A,B\) will depend on the boundary conditions (on the values of \(Z_0, Z_0, V_0\)).

The frequency characteristics are determined by the propagation constant \(-\tau\); the \(-\) sign in \(\tau\) is just convention as you will see.

\[-\tau = \alpha + i \gamma\]

Hence \(A e^\tau = A e^{-\alpha x} e^{-i \gamma y}\)

\(\alpha = \text{attenuation constant}\)
\(\gamma = \text{phase constant}\).

So as the number of loops or \(T\) terms increase, the attenuation increases; the attenuation per section is \(e^{-\alpha x}\) as the current flows towards the right. Similarly the term \(B e^{-n \tau}\) (as you will see from the solution) represents the attenuation as the current flows to the left after it is reflected from the \(Z_2\) end. Hence if \(\alpha > 0\) and there are many sections the current goes to zero at the far end. On the other hand if \(\alpha = 0\) the current flows (is transmitted) freely without any attenuation.
To get the constants $A, B, C, y$ we plug back into the equation
\[-Z_2 \left(n-\frac{1}{2}\right) + \left(Z_1 + 2Z_2\right) \hat{v}_n - Z_2 \hat{v}_{n+1} = 0\]

\[-Z_2 \left(n-\frac{1}{2}\right) + \left(Z_1 + 2Z_2\right) \left[A e^{n\lambda} + B e^{-n\lambda}\right] - Z_2 \left[A e^{n\lambda} + B e^{-n\lambda}\right] = 0\]

or
\[-Z_2 \left(A e^{n\lambda} + A e^{-n\lambda}\right) - (n-\frac{1}{2}) (A e^{n\lambda} + B e^{-n\lambda}) + \left(Z_1 + 2Z_2\right) \left[A e^{n\lambda} + B e^{-n\lambda}\right] = 0\]

\[-Z_2 \left(e^{n\lambda} + e^{-n\lambda}\right) + \frac{1}{2} \left(Z_1 + 2Z_2\right) \left(A e^{n\lambda} + B e^{-n\lambda}\right) = 0\]

or
\[\left(Z_1 + 2Z_2\right) - Z_2 \left(e^n + e^{-n}\right) \left[A e^{n\lambda} + B e^{-n\lambda}\right] = 0\]

We do not want $A e^{n\lambda} + B e^{-n\lambda} = 0$ since this is the current and
leaving the current $= 0$ is not an interesting solution. Hence,
we try
\[e^n + e^{-n} = \frac{Z_1 + 2Z_2}{Z_2} \rightarrow \cosh n = \frac{Z_1 + 2Z_2}{2Z_2}\]

\[H_n = A e^{n\lambda} + B e^{-n\lambda}\]

\[I_n = \left(A e^{n\lambda} + B e^{-n\lambda}\right) e^{i\omega t}\]

Here $A$ and $B$ depend on the boundary conditions (the values
of $Z_0, Z_1$ and $V_0$). The frequency characteristics are determined by
the propagation constant ($\Gamma$)

\[-\Gamma = \alpha + i\gamma \quad \text{- sign is convention}\]

Hence $A e^{n\lambda} = A e^{-\alpha} e^{-n\gamma}$
So as $n$ increases (the number of $T$ or $\pi$ sections increase) there is more of an attenuation in the current given by $e^{-x/n}$ section as the current flows towards the right. Similarly $B e^{-x}$ represents the attenuation as the current flows towards the left after being reflected from the far right, the $Z_2$ end as you will see. Hence if $\delta = 0$ and there are many sections the current goes to zero at the far end. On the other hand if $\delta = 0$ the current is transmitted undiminished. Hence

\[ x = \text{attenuation constant} \]
\[ \phi = \text{phase constant} \]

We talk now about the frequency characteristics of the filter. All filters are made with very small resistance compared to $X_2$ and $X_3$. Hence in our analysis we can assume $Z_1$ and $Z_2$ to be purely imaginary. Hence

\[ \cosh R = \frac{Z_1 + 2Z_2}{2Z_2} = \text{real} \]
\[ \cosh R = \cosh(-\pi) = \cosh(\alpha+i\phi) = \frac{e^{\alpha+i\phi} + e^{-\alpha-i\phi}}{2} \]
\[ \cosh R = (\cosh \alpha + \sinh \alpha)(\cos \phi + i \sin \phi) + (\cosh(-\alpha)(\cos \phi - i \sin \phi) \]
\[ = \cosh \alpha \cos \phi + i \sinh \alpha \sin \phi \]

Since $\cosh R$ is real we have $\sinh \alpha \sin \phi = 0$

Therefore

$\cosh R$ real means $\alpha = 0$ or $\phi = n\pi$ n integer
But \( \cosh \alpha \geq 1 \) always; \( \cos \phi \leq 1 \) always

There are 3 possibilities of interest

\[-1 \leq \cosh \Gamma \leq 1 \quad \text{only possible if } \alpha = 0 \quad \Gamma = -i\phi \text{ or } i\phi\]
\[\cosh \Gamma > 1 \quad \phi = 0 \quad \Gamma = -\alpha \text{ or } \alpha\]
\[\cosh \Gamma < -1 \quad \phi = \pm \pi \quad \Gamma = -\alpha \pm i\pi\]

Now we remember that if \( \alpha = 0 \), the current is allowed to get to the load impedance without attenuation. Hence, the first of the three possibilities is called "the pass band".

\[\alpha = 0 \quad \text{pass band} \quad -1 \leq \frac{Z_1 + 2Z_2}{2Z_2} \leq 1 \quad \text{or } -\frac{Z_1}{Z_2} \leq 0\]

\[\alpha > 0 \quad \text{stop band} \quad \frac{Z_1 + 2Z_2}{2Z_2} > 1 \quad \text{or } <-1\]
\[\frac{Z_1}{Z_2} > 0 \quad \text{or } \frac{Z_1}{Z_2} < -4\]

In the stop band region \( \cosh \alpha = \pm \left\{ \frac{Z_1 + 2Z_2}{2Z_2} \right\} \). This then tells us how the current decays given what \( Z_1 + Z_2 \) are.

Consider a filter made up of the following elements:

---

![Filter Diagram](image-url)
\[ Z_1 = i \omega LC \quad Z_2 = -i \frac{1}{\omega L C} \]
\[ \frac{Z_1}{Z_2} = -\omega^2 LC \]

\[ \cosh \Phi = \cos \alpha \cos \phi = \frac{Z_1}{Z_2} + 1 = 1 - \frac{\omega^2 LC}{2} \]

Hence, the pass band is:
\[ -1 \leq 1 - \frac{\omega^2 LC}{2} \leq 1 \]
\[ 0 \leq 2 - \frac{\omega^2 LC}{2} \leq 2 \]
\[ 0 \leq 4 - \omega^2 LC \leq 4 \]
\[ 0 \leq \omega \leq 2\sqrt{\frac{1}{LC}} \]

The quantity \( \omega_c = \sqrt{\frac{1}{LC}} \) is known as the cut-off frequency. This filter allows small frequencies to go through and is known as a low frequency pass filter.

In the stop band region (see page 104)

\[ \cosh \alpha = \frac{Z_1}{Z_2} + 1 = \pm \left( 1 - \frac{\omega^2 LC}{2} \right) = \left( \omega^2 LC - 1 \right) \frac{1}{2} \]

Note \( \cosh \alpha = 1 \) at \( \omega = \omega_c \) and gets larger as \( \omega \) increases.
Consider a filter made up of different elements as follows:

\[ Z_1 = -\frac{i}{\omega C} \quad Z_2 = i\omega L \]

\[
\cosh \Gamma = \cosh \alpha \cos \phi = \frac{Z_1 + 1}{2Z_2} = 1 - \frac{1}{2\omega^2 LC}
\]

**Pass band** \(-1 \leq \cosh \alpha \cos \phi \leq 1\) \quad \alpha = 0 \quad \cosh \alpha = 1

\[ \omega_c^2 = \frac{1}{4LC} \]

\[ \omega_c = \frac{1}{\sqrt{2LC}} \]

**Cut-off frequency**

\[ \cos \phi = 1 - \frac{1}{2} \omega_c^2 LC \quad \cos \phi = -1 \text{ when } \omega^2 = \frac{1}{4LC} \quad \omega = \omega_c \]

\[ \cos \phi = +1 \text{ when } \omega = \infty \]

This is a high frequency pass band.
We can construct more complicated filters; for example

\[ Z_1 = \frac{i \omega L_1}{1 - \omega^2 L_1 C_1} \text{ from } \frac{1}{Z_1} = \frac{1}{X_L} + \frac{1}{X_C} = \frac{1}{i \omega L_1} + \frac{1}{i \omega C_1} \]

\[ Z_2 = i \omega L_2 - \frac{i}{\omega C_2} = i \left( \frac{\omega^2 L_2 C_2 - 1}{\omega C_2} \right) \]

The band pass condition becomes

\[ 0 \leq \frac{L_1 C_2 \omega^2}{(1 - \omega^2 L_1 C_1)(1 - \omega^2 L_2 C_2)} \leq 4 \]

Gets complicated but essentially this stops propagation between two frequencies. You can exchange \( Z_1 \) and \( Z_2 \) to allow propagation between two frequencies. The possibilities are endless.

**Transmission Characteristics** — Now we want to determine \( A \) or \( B \) which describe some part of the amplitude of \( I \) as a function of the loop number. To do this we have to determine what is known as the "characteristic impedance of the filter". This impedance describes how the circuit behaves if it had an
infinite number of T or Pi sections. We consider the case of a circuit with T sections. Let the characteristic impedance be called $Z_k$.

We recall the general equation

$$i_m = i_{mo} e^{wt}$$

where $V = V_0 e^{jwt}$

$$i_{mo} = A e^{\frac{zm}{m}} + B e^{-\frac{zm}{m}}$$

When we solve for $A + B$ we will see the $B$ term goes to zero when $m$ (the number of sections) is very large. We will see that the $B$ term describes the reflected term and we will see that $B = 0$ when $Z_n = Z_k$ (the characteristic impedance).

Let us calculate what happens when $Z_k$ is the end of the loops or $B = 0$.

$$i_{mo} = A e^{\frac{zm}{m}} = i_{00} e^{\frac{zm}{m}}$$

since $A = i_{00}$

The equation for the $1^{st}$ loop becomes

$$Z_0 i_{00} + \frac{1}{2}Z_1 i_{00} - Z_2 (i_{10} - i_{20}) = V_0$$

The factor $e^{jwt}$ is common on both sides.
Some \( \ell_{10} = 200 \, e^{17} \)

\[
\begin{align*}
Z_0 \, 200 + \frac{1}{2} Z_1 \, 200 & - Z_2 \, (\ell_{10} - 200) = V_0 \\
Z_0 \, 200 + \frac{1}{2} Z_1 \, 200 & - Z_2 \, (e^{17} - 1) \, 200 = V_0
\end{align*}
\]

This must be the same with the characteristic impedance at the end of the 0th loop

\[
(Z_0 + Z_k) \, 200 = V_0
\]

1. Hence \( Z_k = \frac{1}{2} Z_1 - Z_2 \left( e^{17} - 1 \right) \)

We obtain another expression for \( Z_k \). Using our generic equation for \( e^{17} \)

\[
\frac{Z_1}{Z_2} + 1 = e^{17} \frac{e^{17} + e^{17}}{2}
\]

we get

\[
\frac{Z_2 + Z_1}{2} = \frac{Z_2}{2} \left( e^{17} + e^{17} \right)
\]

or

\[
2Z_2 + Z_1 - Z_2 \, e^{-17} = Z_2 \, e^{17}
\]

or,

\[
Z_k = -Z_2 \, e^{17} + \frac{1}{2} Z_1 + Z_2
\]

\[
= -2Z_2 - Z_1 + Z_2 e^{-17} + \frac{1}{2} Z_1 + Z_2
\]

\[
= Z_2 e^{-17} - Z_2 - \frac{1}{2} Z_1
\]

2. \( Z_k = Z_2 \left( e^{-17} - 1 \right) - \frac{1}{2} Z_1 \)

We continue obtaining new expressions for \( Z_k \). Using 1

\[
Z_k = -Z_2 \, e^{17} + \frac{1}{2} \left( Z_1 + 2Z_2 \right)
\]
\[
\frac{1}{Z_{\text{par}}} = \frac{1}{Z_2} + \frac{1}{\frac{1}{2}Z_1 + Z_k} = \frac{\frac{1}{2}Z_1 + Z_2 + Z_k}{Z_2 \left( \frac{1}{2}Z_1 + Z_k \right)}
\]

\[
Z_{\text{par}} = \frac{\frac{1}{2}Z_1 Z_2 + Z_k Z_2}{\frac{1}{2}Z_1 + Z_2 + Z_k}
\]

\[
Z_? = \frac{1}{2}Z_1 + Z_{\text{par}} = \frac{1}{2}Z_1 + \frac{1}{2}Z_1 Z_2 + Z_k Z_2
\]

\[
Z_? (\frac{1}{2}Z_1 + Z_2 + Z_k) = \frac{1}{2} Z_1^2 + \frac{1}{2} Z_1 Z_2 + \frac{1}{2} Z_2 Z_k + \frac{1}{2} Z_k Z_2 + Z_k Z_2
\]

\[
= \frac{1}{2} Z_1^2 + Z_1 Z_2 + Z_k (\frac{1}{2} Z_1 + Z_2)
\]

\[
= Z_k + Z_k \left( \frac{1}{2} Z_1 + Z_2 \right)
\]

where we replace \( Z_k = \left( Z_1^2 + Z_2 Z_2 \right)^{\frac{1}{2}} \)

Hence we have

\[
Z_? (\frac{1}{2}Z_1 + Z_2 + Z_k) = Z_k (Z_k + \frac{1}{2} Z_1 + Z_2)
\]

or

\[
Z_? = Z_k
\]

It turns out that it does not matter how many \( k \) elements you have in between you always get \( Z_? = Z_k \), namely the current in each element \( Z_0, Z, \ldots \) acts as if you had an infinite number of elements.

Now we proceed to solve for A and B and understand the meaning of \( Z_k \).
We write the equations for the T-type filter. We have for the 0th loop and the last loop (labelled 1):

1. \( Z_0 I_{00} + \frac{1}{2} Z_1 I_{10} - Z_2 (I_{10} - I_{00}) = 0 \) where \( I_n = I_{n0} e^{j\omega n} \)
2. \( -Z_2 I_{(L-1)0} + (Z_2 + \frac{1}{2} Z_l) I_{L0} + Z_n I_{L0} = 0 \)

Now \( I_{n0} = A e^{jn\omega} + Be^{-jn\omega} \)
\( I_{00} = A + B \)
\( I_{10} = A e^{jn\omega} + B e^{-jn\omega} \)
\( I_{L0} = A e^{jn\omega L} + B e^{-jn\omega L} \)
\( I_{(L-1)0} = A e^{jn\omega (L-1)} + B e^{-jn\omega (L-1)} = A e^{jn\omega L} + B e^{-jn\omega L} \)

Substituting into 1:

\( (Z_0 + \frac{1}{2} Z_1 + Z_2)(A + B) - Z_2 (A e^{jn\omega} + B e^{-jn\omega}) = V_0 \)

or \( [-Z_2 (e^{jn\omega -1}) + \frac{1}{2} Z_1 + Z_0] A + [-Z_2 (e^{-jn\omega -1}) + \frac{1}{2} Z_1 + Z_0] B = V_0 \)

Substituting into 2:

\( [-Z_2 (e^{jn\omega L} + (Z_2 + \frac{1}{2} Z_l) e^{jn\omega L})] A + [-Z_2 e^{jn\omega} + (Z_2 + \frac{1}{2} Z_l) e^{-jn\omega} + Z_n e^{-jn\omega}] B = 0 \)

or \( e^{jn\omega L} [-Z_2 (e^{jn\omega -1}) + \frac{1}{2} Z_1 + Z_n] A - e^{-jn\omega L} [-Z_2 (e^{-jn\omega -1}) + \frac{1}{2} Z_1 + Z_n] B = 0 \)

Using equation 2 for the characteristic impedance \( Z_k \):

3. \( [Z_k + Z_0] A + [-Z_k + Z_0] B = V_0 \)
4. \( e^{jn\omega L} [-Z_k + Z_n] A + e^{-jn\omega L} [Z_k + Z_n] B = 0 \)
Solving for \( A + B \). From (3)

\[
A = \frac{V_0 + [Z_k-Z_n]B}{Z_k+Z_n}
\]

Substituting into (4)

\[
-e^{nl} (Z_k-Z_n) \left[ \frac{V_0 + (Z_k-Z_n)B}{Z_k+Z_n} \right] + e^{-nl} (Z_k+Z_n) B = 0
\]

Dividing by \( Z_k+Z_n \) and defining \( T_L = \frac{Z_k-Z_n}{Z_k+Z_n} \), \( T_0 = \frac{Z_k-Z_0}{Z_k+Z_0} \)

\( T_L, T_0 \) the reflection coefficients at the end and beginning of the filter

we get

\[
-e^{nl} T_L \left[ \frac{V_0}{Z_k+Z_n} \right] + B \left[ e^{-nl} - e^{nl} T_L T_0 \right] = 0
\]

(5)

\[
B = \frac{V_0 e^{nl}}{Z_k+Z_n} \frac{T_L e^{nl}}{e^{-nl} - e^{nl} T_L T_0}
\]

\[
B = \frac{V_0}{Z_k+Z_n} \frac{T_L e^{2nl}}{1 - e^{2nl} T_L T_0}
\]

note \( B = 0 \) when \( T_L = 0 \) \((Z_k = Z_n)\)

Note eq (4) can be written

\[
A T_L e^{nl} = B e^{-nl}
\]

\[
A = B e^{-2nl} T_L
\]
Hence we can write

\[
R = \frac{V_0}{Z_k + Z_0} \frac{1}{1 - e^{2\pi L T_k T_0}}
\]

Hence we get the final equation for the current

\[
I_{no} = -\frac{V_0}{Z_k + Z_0} \frac{e^{\pi n + T_L e^{(2\pi L - \pi n)}}}{1 - e^{2\pi L T_k T_0}}
\]

\[
I_n = \frac{V_0}{Z_k + Z_0} \frac{e^{\pi n + T_L e^{(2\pi L - \pi n)}} e^{i\omega t}}{1 - e^{2\pi L T_k T_0}}
\]

where \( R = -\alpha - i\beta \)

\[
\cosh R = \frac{Z_1 + 2Z_2}{2Z_2}
\]

We can take the denominator and expand it and we get

\[
I_{no} = \frac{V_0}{Z_k + Z_0} \left[ e^{\pi n + T_L e^{(2\pi L - \pi n)}} \left[ 1 + e^{2\pi L T_k T_0} e^{\pi (2\pi L - \pi n)} (T_k T_0)^2 + \ldots \right] \right]
\]

\[
= \frac{V_0}{Z_k + Z_0} \left\{ e^{\pi n + T_L e^{(2\pi L - \pi n)}} + e^{\pi (2\pi L - \pi n)} T_k T_0 + e^{\pi (4\pi L - \pi n)} (T_k T_0)^2 + \ldots \right\}
\]

Hence, the interpretation of \( T_k \) and \( T_L \) as the reflection coefficient.

Note \( T_k > 0 \) if \( Z_k < Z_0 \) reflection in phase.

Similarly for \( T_k \) out of phase.
We can do the same analysis for \( n \) type filters.

The equations for the first few loops and last two loops:

\[
\begin{align*}
Z_0 I_0 + 2Z_2 (I_0 - I_1) &= V_0 \\
2Z_2(I_1 - I_0) + Z_1 I_1 + Z_2 (I_1 - I_2) &= 0 \\
Z_2 (I_2 - I_1) + Z_1 I_2 + Z_2 (I_2 - I_3) &= 0
\end{align*}
\]

The general equation:

\[
Z_2 (I_n - I_{n-1}) + Z_1 I_n - Z_2 (I_{n+1} - I_n) = 0 \\
(Z_1 + 2Z_2) I_n - Z_2 I_{n+1} - Z_2 I_{n-1} = 0
\]

The last two loops:

\[
\begin{align*}
Z_2 (I_{L-2} - I_L) + Z_1 I_{L-1} + Z_2 (I_{L-1} - I_{L-2}) &= 0 \\
I_L Z_n + 2Z_2 (I_L - I_{L-1}) &= 0
\end{align*}
\]

The general expression is the same as for the \( T \) type filters. We need to fix \( I_{10} \) and \( I_{00} \) and similarly \( I_{L-1} \) and \( I_L \). We need to rewrite:

\[
\begin{align*}
Z_0 I_{10} + 2Z_2 (I_{10} - I_1) - V_0 &= -Z_2 I_{10} + Z_1 + 2Z_2 I_{10} + Z_2 I_{10} \\
\text{or } Z_2 I_{10} &= V_0 + 2Z_2 I_{10} - Z_0 I_{10} + Z_1 I_{10} + Z_2 I_{10}
\end{align*}
\]

Better is to replace \( Z_2 (I_{10} - I_{00}) \) with \( Z_2 (I_{10} - I'_{00}) \), then we have:

\[
- Z_2 I_{00} + (2Z_2 + Z_1) I_{10} - Z_2 I_{20} = 0
\]

which agrees with the general eq. for the \( n=1 \) loop.
Then \( I'_{00} = 2I_{00} - I_{10} \)

Then the \( n=0 \) loop becomes

\[
\begin{align*}
Z_0 I_{00} + Z_2 (I_{10} - I'_{00}) - V_0 &= 0 \\
- \frac{Z_0}{2} I_{10} + \frac{Z_2}{2} (2I_{00} - I_{10}) + V_0 &= 0 \\
- \frac{Z_0}{2} I'_{00} + \frac{Z_0}{2} I_{10} + \frac{Z_2}{2} (I_{10} - I'_{00}) + V_0 &= 0 \\
- \left( \frac{Z_0}{2} + Z_2 \right) I''_{10} + \left( - \frac{Z_0}{2} + Z_2 \right) I_{10} + V_0 &= 0 \\
&\text{or} \left( \frac{Z_0}{2} + Z_2 \right) I''_{00} + \left( \frac{Z_0}{2} - Z_2 \right) I_{10} = V_0
\end{align*}
\]

Now \( I_{10} = I_{00} e^r = 2I_{00} e^r - I_{10} e^r = 2I_{00} e^r - I_{00} e^{2r} \)

\( = 2I_{00} e^r - I_{10} e^r \)

\( I_{10} (1 + e^r) = 2I_{00} e^r \)

\( I_{10} = I_{00} 2e^r (1 + e^r) \)
\[ Z_0 I_0 + 2Z_2 (I_0 - I_1) = V_0 \]
\[ 2Z_2 (I_1 - I_0) + Z_1 I_1 + Z_2 (I_1 - I_2) = 0 \]
\[ Z_2 (I_2 - I_1) + Z_1 I_2 + Z_2 (I_2 - I_3) = 0 \]
\[ -Z_2 I_{n-1} + (Z_1 + 2Z_2) I_n - Z_2 I_{n+1} = 0 \quad \text{General eq.} \]
\[ I_n = I_{n0} e^{j\omega t} \]
The characteristic impedance comes from

\[ (Z_0 + 2Z_2) I_0 - 2Z_2 (I_0 e^{j\pi}) = (Z_0 + Z_k) I_0 = V_0 \]

\[ Z_k = 2Z_2 (1 - e^{j\pi}) \]

\[ \cosh \, \pi = \frac{Z_1}{2Z_2} + 1 \]
\[ \frac{e^{j\pi} + e^{-j\pi}}{2} = \frac{Z_1}{2Z_2} + 1 \quad \text{or} \quad \frac{e^{j\pi} - e^{-j\pi}}{2Z_2} = \frac{Z_1}{Z_2} + \frac{1}{2} \]
\[ 2 - e^{j\pi} = e^{-j\pi} - \frac{Z_1}{Z_2} \quad \text{or} \quad 1 - e^{j\pi} = e^{-j\pi} - \frac{Z_1}{Z_2} \]
\[ 2Z_2 (1 - e^{j\pi}) = 2Z_2 (e^{-j\pi} - 1 - \frac{Z_1}{Z_2}) \]
2Z_k = 2Z_2(e^{-n} - 1) - 2Z_1 + 2Z_2(1 - e^n)
    = 2Z_2(e^{-n} - e^n) - 2Z_1 + 2Z_2

(Z_k + Z_1 - Z_2) = Z_2(e^{-n} - e^n)

(Z_k + Z_1 - Z_2)^2 = 4Z_2^2 \left(\frac{e^{-n} - e^n}{2}\right)^2

    = 4Z_2^2 \left[\left(\frac{e^n + e^{-n}}{2}\right)^2 - 1\right]
    = 4Z_2^2 \left[\left(\frac{Z_1 + 1}{2Z_2}\right)^2 - 1\right]
    = Z_1^2 + 4Z_2Z_1

Z_k^2 + 2Z_kZ_1 - Z_1Z_2 = Z_2^2 + 6Z_2Z_1

\frac{e^n + e^{-n}}{2} = \frac{Z_1 + 1}{2Z_2}

\frac{e^n + e^{-n}}{2} = \frac{Z_1 + 1}{Z_2}

\frac{e^n + e^{-2n}}{2} = \frac{Z_1}{Z_2} \left(\frac{1}{e^n} + e^{-n}\right)

1 - e^{-n} = \frac{Z_1}{Z_2} e^{-n} - e^{-2n}
\[ V_0 = (Z_0 + Z_{k}) \left( I_0 + I_1 \right) \]
\[ = Z_0 \left( I_0 + I_1 \right) Z Z_2 (I_0 - I_1) \]

\[ Z_k = 2 Z_2 \left( 1 - \frac{e^n}{1 + e^n} \right) \]

\[ = 2 Z_2 \left( 1 + \frac{e^n - e^{-n}}{1 + e^n} \right) = \frac{2 Z_2 (1 - e^{-n})}{1 + e^n} = 2 Z_2 \frac{e^{-n/2} - e^{n/2}}{e^{-n/2} + e^{n/2}} \]

\[ \frac{e^n + e^{-n}}{2} = \frac{Z_2}{2} \left( \frac{1}{1 + e^{n/2}} \right) \]

\[ \frac{e^n}{2 e^{n/2}} = \frac{Z_2}{2} \left( \frac{1}{1 + e^{n/2}} \right) \]

\[ Z_k = \frac{4 Z_2}{1 + 2 e^n + 2 e^{-n}} \]

\[ = \frac{4 Z_2^2 \left( e^n e^{-n} - 2 \right)}{e^n e^{-n} + 2} \]

\[ = 4 Z_2^2 \left\{ \frac{Z_1/Z_2}{Z_1 + 4} \right\} = 4 \frac{Z_2^2 Z_1}{Z_1 + 4 Z_2} \]
Transmission Lines

So far we have dealt with finite elements and the current has a constant value in each loop so that the current changes discontinuously from one loop to the next. We now consider the case where the elements of the current are distributive and the current is changing continuously both in distance along the circuit and in time.

\[ Z_0 = \text{Generator Impedance} \]
\[ Z_n = \text{Load Impedance (like the factory or house)} \]

The line has an inductance \( L \) and resistance \( R \) per unit length and there is a capacitance per unit length \( C \) between the central wire and the outer conductor. So we have a resistance and inductance per unit length along the central conductor \( Z_S = R + i \omega L \) and a capacitance/unit length in parallel \( Z_P = -i/\omega C \).

\[ \frac{dI}{ds} = I_3 - dI_5 \quad dI = \frac{dI_5}{ds} \quad ds \]

Since \( dI_5 \to 0 \) we write \( |dI| = -\frac{dI_5}{ds} \)
\[ V_s = X_e |dI| - \frac{i}{\omega C} |dI| \]

Hence \( \frac{dV_s}{ds} = 1 \frac{dI}{ds} = \frac{V_s}{X_e} - i \frac{\omega C ds}{\omega C} ds \)

\[ \frac{dV_s}{ds} = \frac{V_s}{Z_p} ds \]

Now \( V_s \) decreases as you go along the wire a distance \( ds \) because of the series impedance.

\[ \frac{\partial V_s}{\partial s} = -Z_s I_s \]

From equation 1 we have \( \frac{\partial^2 I_s}{\partial s^2} = -\frac{1}{Z_p} \frac{\partial V_s}{\partial s} \)

Applying eq 2 \( \frac{\partial^2 I_s}{\partial s^2} = +\frac{Z_s}{Z_p} I_s \) Hence

\[ \frac{\partial^2 I_s}{\partial s^2} = \frac{Z_s}{Z_p} I_s \]

The diff. equation of a transmission line.

Note that this does not include the time dependence which by now we know it is the time dependence of the generator = \( e^{jut} \)
So we can write

$$I_5 = I_{so} e^{i\omega t}$$

$$\frac{d^2 I_{so}}{ds^2} - \frac{Z_s}{Z_p} I_{so} = 0$$

The solution of this differential equation is

$$I_{so} = A e^{rs} + B e^{-rs}$$

where $-r = \alpha + i \beta$ is the propagation constant.

So this looks just like the filter problem except that $s$ is now a continuous variable. $A$ and $B$ could be complex, so let $A = Ae^{ia}$, $B = Be^{ib}$

$$I_5 = A e^{-\alpha s} e^{i(\omega t - ps + a)} + B e^{\alpha s} e^{i(\omega t + ps + b)}$$

and the real part (if $V = V_0 \cos \omega t$)

$$I_5 = A e^{-\alpha s} \cos(\omega t - ps + a) + B e^{\alpha s} \cos(\omega t + ps + b)$$

wave traveling away from $V$, reflected wave traveling towards larger $s$, back towards $V$.

Now we interpret differently

$$\lambda = 2\pi / \gamma$$

$$u = \lambda f = \lambda \omega = \frac{\omega}{2\pi}$$

phase velocity

Now, trying substituting our solution into the diff. equation

$$\nabla^2 (A e^{i\beta s} + B e^{-i\beta s}) = \frac{Z_s}{Z_p} (A e^{i\beta s} + B e^{-i\beta s})$$
Hence \( \Gamma = \pm \sqrt{\frac{Z_0}{Z_p}} \)

We keep the negative solution because \( \Gamma = -(\alpha + i\beta) \)
\( \Gamma = -\sqrt{\frac{Z_0}{Z_p}} \)

We check how or why this differs from the filter equation

\[
\frac{e^{\Gamma} + e^{-\Gamma}}{2} = \frac{Z_1 + 2Z_2}{2Z_2} = \frac{Z_0 + 2Z_p}{2Z_p}
\]

In the treatment of transmission lines we use differentials and only keep the first order. Hence \( Z_0 \ll \text{and} \ Z_p \gg \) or \( Z_0/Z_p \ll 1 \). If we had such a filter made of finite elements we can expand

\[
\frac{e^{\Gamma} + e^{-\Gamma}}{2} = (1 + \Gamma + \frac{\Gamma^2}{2!} + \frac{\Gamma^3}{3!} + \frac{\Gamma^4}{4!}) - (1 - \Gamma - \frac{\Gamma^2}{2!} - \frac{\Gamma^3}{3!} - \frac{\Gamma^4}{4!})
\]

\[
= 1 + \frac{\Gamma^2}{2} + \frac{\Gamma^4}{4} = \frac{Z_1}{2Z_2} + 1 = \frac{Z_0}{2Z_p} + 1
\]

or \( \Gamma = -\sqrt{\frac{Z_0}{Z_p}} = -\sqrt{\frac{Z_0}{Z_p}} \)

which is the same answer for \( \Gamma \ll 1 \)

We now can guess that everything is the same as for filters in the limit \( Z_2 \gg Z_1 \). From equation \( \theta \) in page \( 10 \)

\[
Z_R = \left[ Z_1^2 + Z_1Z_2^3 \right]^{\frac{1}{2}} = \sqrt{Z_1Z_2} = \sqrt{i\omega L - \frac{1}{C}} = i\sqrt{\frac{Z_0}{Z_p}} \quad \text{if} \quad R = 0
\]

\[
Z_1 = R + i\omega L = i\omega L
\]
Now we go further and solve for $\alpha$ and $\varphi$

$$I^2 = \frac{Z_0}{Z_p} \cdot (R + j\omega L) i \omega C$$

$$(\alpha + j\varphi)^2 = -\omega^2 LC + jR \omega C$$

$$\alpha^2 - \varphi^2 = -\omega^2 LC$$

$$2\alpha \varphi = R \omega C$$

Solving for $\alpha$

$$\alpha^2 - \frac{R^2 \omega^2 C^2}{4\alpha^2} = -\omega^2 LC$$

$$4\alpha^4 + 4\alpha^2 \omega^2 LC - R^2 \omega^2 C^2 = 0$$

Let $\alpha' = \alpha^2$ solve for $\alpha'$ and thus for $\alpha = \sqrt{\alpha'}$

$$\alpha = \left\{ -\frac{\omega^2 LC + \sqrt{\omega^2 C^2 (L^2 \omega^2 + R^2)}}{2} \right\}^{\frac{1}{2}}$$

$$\varphi = \alpha^2 + \omega^2 LC = \frac{\omega^2 LC + \sqrt{\omega^2 C^2 (L^2 \omega^2 + R^2)}}{2}$$

$$\varphi = \left\{ \frac{\omega^2 LC + \sqrt{\omega^2 C^2 (L^2 \omega^2 + R^2)}}{2} \right\}^{\frac{1}{2}}$$

In a good transmission line, as opposed to a good filter, you want a propagation constant which is independent of the frequency since you want all frequencies to be propagated equally. In the case for telephone propagation, computer information, signal distortion before it is observed by a scope, etc. To see how we achieve this, give write
\[ \alpha = \left\{ -\frac{w^2LC}{2} + \frac{w^2LC}{2} \left[ 1 + \frac{R^2}{w^2L^2} \right]^{1/2} \right\}^{1/2} \]

If we can make \( R/wL \ll 1 \), which is a good reason for superconducting transmission lines.

\[ \alpha = \left\{ -\frac{w^2LC}{2} + \frac{w^2LC}{2} \left[ 1 + \frac{1}{2} \frac{R^2}{w^2L^2} - \frac{1}{8} \frac{R^4}{w^4L^4} \ldots \right] \right\}^{1/2} \]

\[ \alpha = \left\{ \frac{1}{4} \frac{R^2C}{L^2} \right\}^{1/2} = \frac{1}{2} \frac{R}{\sqrt{C}} \]

\[ \phi = \left\{ \frac{w^2LC + \alpha^2}{4} \right\} = \left\{ \frac{w^2LC}{4} + \frac{R^2C}{L} \right\}^{1/2} = \sqrt{\frac{w^2LC}{4}} \]

\[ \nu = \frac{1}{\phi \sqrt{LC}} \]

\[ \frac{1}{\sqrt{LC}} \]

for typical transmission lines turns out to be equal to the velocity of light \( c \).

Now it is usually hard to reduce \( R \) below the needed value for a copper transmission line, so what one does is to increase \( L \) by adding small inductance coils at regular coil intervals. This is called a loaded line. This you can see in lines along highways. Also note that if \( L \) is \( \gg \alpha \) is small.