\[ \Phi = \oint \vec{B} \cdot d\vec{s} = 4\pi N \pi R^2 \frac{I}{L} \quad R = \text{radius of solenoid} \]

This is the flux through one loop of the solenoid

\[ L = \frac{\Phi_{\text{total}}}{I} = \frac{N \Phi}{I} \quad N = \text{total number of loops} \]

\[ = 4\pi N^2 \pi R^2 L \quad \pi R^2 L = \text{volume of solenoid} \]

\[ = 4\pi n^2 V \quad n = \text{number of turns/unit length} \]

We can carry out a similar calculation to discuss coefficients of capacitance. We can determine self-capacitance and mutual capacitance.

Consider a system of conducting surfaces each with a charge \( q_i \) and a potential \( \Phi_i = V_i \). We determined a while ago

\[ U = \frac{1}{2} \sum q_i V_i \]

\[ V_i = \sum_j \left( \frac{q_j}{\sqrt{|r_i - r_j|^2}} \right) \quad \text{where} \quad j \neq i \]

\[ C_{ij} = \sum_j \left( \frac{1}{|r_i - r_j|^2} \right) d^3 r' = \frac{1}{2} \frac{V_i}{V_j} \]

Now if \( \rho_j = \text{const.} \) and on the surface of a conductor

\[ \rho_j d^3 r' = \oint \sigma d^2 s' \]

Let us consider the case of a parallel conducting plate with a charge/unit area = \( \sigma \) on the surface of the plates
We calculated long ago
\[ E = 4\pi \sigma = 4\pi \frac{Q}{A} \]
\[ V = EL = 4\pi \frac{Q}{A} \approx \frac{Q}{C} \]
\[ C = \left(4\pi \frac{E}{A}\right)^{-1} \]

This is the case for the self-capacitance

**Circuit Theory**

Inductors, Capacitors, and Resistors use up energy and a voltage source like batteries provide energy. The equation of a circuit is just energy conservation or voltage balance. We use MKS units because all circuit elements are in MKS.

**Source of potential** \( V(t) \) in voltage = volts
We use up potential in an inductor \( V_L = L \frac{di}{dt} \) in henries
\[ \frac{di}{dt} \text{ if } L \text{ is constant } \]
We use up potential in a capacitor \( = \frac{Q}{C} \) Q = coulombs
\[ \frac{Q}{C} = \text{farads} \]
We use potential in a resistor \( = i R \) i = amperes
\[ i \text{ in amperes } \]
\[ R \text{ in ohms } \]

Consider the circuit below in various situations where we provide energy and see what happens under certain conditions.

We initially put a voltage in the capacitor \( = \frac{Q}{C} = V_0 \) and leave switch 1 open and close switch 2.
We call $t=0$ when we close switch 2. The batteries are left out of the circuit because no current flows there. The voltage equation becomes

$$\frac{L}{dt} \frac{di}{dt} + Ri + \frac{Q}{C} = 0$$

Recalling $i = \frac{dQ}{dt}$ we get

$$\frac{L}{dt^2} \frac{d^2Q}{dt^2} + \frac{R}{dt} \frac{dQ}{dt} + \frac{Q}{C} = 0$$

This is a 2nd order differential equation and the solutions are of the form which must hold for all $t$.

$$Q = Ae^{-\alpha t}$$

Clearly $A=0$ is a solution but is not an interesting because it implies $Q=0$ at all times. We neglect this solution. To get a solution for $\alpha$ it must be

$$\left\{ L \alpha^2 - R \alpha + \frac{1}{C} \right\} A e^{-\alpha t} = 0$$

Hence another solution, a more interesting one, is that

$$L \alpha^2 - R \alpha + \frac{1}{C} = 0$$

$$\alpha = R/2L \pm \sqrt{R^2/4L^2 - \frac{1}{LC}} = R/2L \pm \eta$$
Hence the more general interesting solution

\[ Q = e^{-\frac{R}{2L}t} \left( A' e^{\eta t} + B' e^{-\eta t} \right) \]

where we have 2 arbitrary constants as is usually required by a 2nd order differential equations. Hence we need two initial conditions at \( t=0 \) which are

\[ Q = Q_0 \text{ at } t=0 \]

\[ i = \frac{dQ}{dt} = 0 \text{ at } t=0 \]

The motion of \( Q \) or \( i \) (time dependence) has 3 types of behavior depending on the character of \( \eta \)

1) \( \frac{1}{2}c > \frac{R^2}{4L^2} \)  \( \eta = i\omega_0 \omega_0 = \left\{ \frac{1}{2}c - \frac{R^2}{4L^2} \right\}^{1/2} \)
   known as underdamped solution

2) \( \frac{1}{2}c = \frac{R^2}{4L^2} \)  \( \eta = 0 \)  known as the critically damped solution. We revisit this case

3) \( \frac{1}{2}c < \frac{R^2}{4L^2} \)  \( \eta \) is real  Overdamped solution

Case 1: Underdamped solution

\[ Q = e^{-\frac{R}{2L}t} \left( A' e^{i\omega_0 t} + B' e^{-i\omega_0 t} \right) \]

To determine \( A + B \) we apply the initial conditions
\[ Q = Q_0 = A' + B' \]
\[ \frac{dQ}{dt} = -R \frac{2L}{L} (A'e^{i\omega t} + B'e^{-i\omega t}) \quad \bigg|_{t=0} \]
\[ + i\omega_0 e^{-R \frac{2L}{L} t} (A'e^{i\omega_0 t} - B'e^{-i\omega_0 t}) \quad \bigg|_{t=0} \]
\[ 0 = -\frac{R}{2L} (A' + B') + i\omega_0 (A' - B') \]
\[ = (-\frac{R}{2L} + i\omega_0) A' - (\frac{R}{2L} + i\omega_0) B' \]

Using \( A' = Q_0 - B' \) we get

\[ 0 = (-\frac{R}{2L} + i\omega_0) Q_0 + (\frac{R}{2L} - i\omega_0) B' - (\frac{R}{2L} + i\omega_0) B' \]

\[ B' = \frac{(-\frac{R}{2L} + i\omega_0) Q_0}{2i\omega_0} \]

\[ A' = Q_0 - B' = Q_0 \left( \frac{2i\omega_0 + \frac{R}{2L} - i\omega_0}{2i\omega_0} \right) \]

\[ A' = \frac{\frac{R}{2L} + i\omega_0}{2i\omega_0} Q_0 \]

\[ Q = Q_0 e^{-R \frac{2L}{L} t} \left\{ \left( \frac{\frac{R}{2L} + i\omega_0}{2i\omega_0} \right) e^{i\omega_0 t} - \left( \frac{\frac{R}{2L} - i\omega_0}{2i\omega_0} \right) e^{-i\omega_0 t} \right\} \]

A complex quantity \( e^{i\theta} = \cos \theta + i\sin \theta = (\frac{\alpha^2 + \beta^2}{\alpha^2 + \beta^2}) e^{i\theta} \)

\[ \alpha + i\beta = \sqrt{\alpha^2 + \beta^2} e^{i\theta} = \sqrt{\alpha^2 + \beta^2} \left( \frac{\alpha + i\beta}{\sqrt{\alpha^2 + \beta^2}} \right) \]

\[ \alpha - i\beta = \sqrt{\alpha^2 + \beta^2} e^{-i\theta} \quad \tan \delta = \beta/\alpha \]

\[ Q = \frac{Q_0}{\omega_0} \frac{\sqrt{\frac{R^2}{4L^2} + \omega_0^2}}{2i} \left\{ \left( \frac{\frac{R}{2L} + i\omega_0}{\sqrt{\left( \frac{\frac{R}{2L} + i\omega_0}{\omega_0} \right)^2 + \omega_0^2}} \right) e^{i\omega_0 t} - \left( \frac{\frac{R}{2L} - i\omega_0}{\sqrt{\left( \frac{\frac{R}{2L} - i\omega_0}{\omega_0} \right)^2 + \omega_0^2}} \right) e^{-i\omega_0 t} \right\} e^{-\frac{R}{2L} t} \]
\[ Q = \frac{Q_0 \sqrt{(R/2L)^2 + w_0^2}}{w_0} \left\{ e^{i(w_0 t + \delta)} - e^{-i(w_0 t + \delta)} \right\} e^{-R/2L t} \]

\[ \tan \delta = \frac{W_0}{\frac{R}{2L}} \quad \cos \delta = \frac{R/2L}{\sqrt{(R/2L)^2 + W_0^2}} \quad \sin \delta = \frac{W_0}{\sqrt{(R/2L)^2 + W_0^2}} \]

\[ \frac{R/2L + iW_0}{\sqrt{(R/2L)^2 + W_0^2}} = \cos \delta + i \sin \delta = e^{i \delta} \]

\[ Q = \frac{Q_0 \sqrt{(R/2L)^2 + W_0^2}}{w_0} \sin(w_0 t + \delta) e^{-R/2L t} \]

\[ = \frac{Q_0}{\sin \delta} e^{-R/2L t} \sin(w_0 t + \delta) \]

Note that if \( R = 0 \) \( \delta = 90^\circ \) because \( \sin 90^\circ = 1 \)

\[ Q = Q_0 \cos w_0 t \]

Plugging this solution into the differential equation:

\[-Lw_0^2 Q_0 \cos w_0 t - R W_0 Q_0 \sin w_0 t + \frac{Q_0 \cos w_0 t}{C} = 0\]

From page 19 case 1 we have \( w_0 = \left\{ \frac{1}{2}c - \frac{R^2}{2L^2} \right\}^{1/2} \)

\[ = \sqrt{\frac{1}{2}c} \quad \text{if} \quad R = 0 \]

This is known as the natural frequency of the circuit.
Note that at $t=0$

$$Q = \frac{Q_0 \sqrt{R^2/A^2 + W_0^2}}{W_0} \sin \theta = Q_0$$

since $\sin \theta = \frac{W_0}{\sqrt{R^2/A^2 + W_0^2}}$

$e^{-\theta R/A t}$ is the damping factor

\[ Q \]
\[ t \]

Description of what is going on

$$Q = Q_0 \cos \omega t$$
$$\dot{Q} = -Q_0 \omega_0 \sin \omega_0 t$$
$$V_L = L \frac{dI}{dt} = -L \omega_0 \omega_0 \cos \omega_0 t = -\frac{Q}{C}$$
$$V_C = \frac{Q}{C}$$
$$\therefore V_L = -V_C$$

Capacitor starts discharging and $\dot{I}/I$ starts producing a voltage across the inductor $L$ which opposes the voltage across the capacitor since it is negative by Lenz's law. Hence the process begins to reverse finally and begins to charge the capacitor and then the process starts over again when the capacitor is fully charged.
The only energy dissipative element is the resistor; hence the term $e^{-\frac{R}{2}t}$.

Case (2) Critically Damped. $\frac{R^2}{4L^2} = \frac{1}{2C}$

Our previous solution would have $\eta = 0$

$$Q = (A + Bt) e^{-\frac{R}{2L}t} = Q_0 e^{-\frac{R}{2L}t}$$

but we know that the most general solution to a second order differential equation must have two arbitrary constants. Hence this cannot be the most general solution.

We have

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0$$

we show that if $\frac{1}{2C} = \frac{R^2}{4L^2}$ the solution turns out

$$Q = (A + Bt) e^{-\frac{R}{2L}t}$$

To show that this is a solution we substitute back

$$L \alpha^2 A e^{-\alpha t} - R A e^{-\alpha t} + A/C e^{-\alpha t} + L(-\alpha B e^{-\alpha t} - \alpha B e^{-\alpha t} + \alpha^2 B^t e^{-\alpha t}) + R(B e^{-\alpha t} - \alpha B^t e^{-\alpha t}) + \frac{1}{C} B e^{-\alpha t} = 0$$

Hence

$$(\alpha^2 L - RA + \frac{1}{C})(A e^{-\alpha t} + B t e^{-\alpha t}) - (2L - R)B e^{-\alpha t} = 0$$
We note this is also a solution if $\alpha = \frac{R}{2L}$ and
\[ \alpha^2 L - R\alpha + \frac{1}{C} = 0 \quad \alpha = \frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{R}{2L} \]
in this case.

Hence $Q(t=0) = A' = Q_0$
\[ i = \frac{dQ}{dt} = (A'e^{\alpha t} + B'e^{-\alpha t})e^{-\alpha t} \]
\[ i = 0 = -A'e^0 + B'e^0 \text{ at } t=0 \]
\[ B = Q_0 \alpha \]

Hence the solution is $Q = Q_0(1 + \alpha t)e^{-\alpha t} \quad \alpha = \frac{R}{2L}$

Case 3, overdamped case \( \frac{R^2}{4L^2} > \frac{1}{LC} \)

The solution is $Q = e^{-\frac{R}{2L} t} (A'e^{nt} + B'e^{-nt}) \quad n = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

Note that this solution term $e^{-\frac{R}{2L} t} \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} t$ which does not decrease as fast as $e^{-\frac{R}{2L} t}$.

It turns out that the fastest way to discharge a circuit is by using the critically damped array.

To solve for $A'$ and $B'$ we have
\[ Q = Q_0 = A' + B' \text{ at } t=0 \]
\[ i = \frac{dQ}{dt} \bigg|_{t=0} = 0 = -\frac{R}{2L} (A+B) + \eta (A-B) \]

The solution is $A = \frac{Q_0}{2\eta} (\eta + \frac{R}{2L})$
\[ B = \frac{Q_0}{2\eta} (\eta - \frac{R}{2L}) \]

These solutions are the transient solutions. The particular solutions if you have the battery included are known as the steady state solutions which are next.
You have a source voltage \( V = V_0 \cos \omega t \) \( (\omega = 2\pi f) \) where \( f \) is the frequency of the source. Consider a circuit which consists of a resistance \( R \), inductance \( L \) and capacitance \( C \) in series with the voltage source:

\[
\begin{array}{c}
R \\
\downarrow \\
L \\
\downarrow \\
C
\end{array}
\]

The voltage differential equation is:

\[
L \frac{dI}{dt} + RI + \frac{Q}{C} = V_0 \cos \omega t \quad I = \text{current}
\]

\( i \to I = \text{change of notation to avoid confusion with } i = V - I \)

To solve problems with \( \cos \omega t \) or \( \sin \omega t \) driving voltage we use a trick. We write 2 equations:

\[
L \frac{d^2Q_x}{dt^2} + R \frac{dQ_x}{dt} + \frac{Q_x}{C} = V_0 \cos \omega t
\]

\[
\left\{ \begin{array}{c}
L \frac{d^2Q_y}{dt^2} + R \frac{dQ_y}{dt} + \frac{Q_y}{C} = iV_0 \cos \omega t \\
i \left[ L \frac{d^2Q_y}{dt^2} + R \frac{dQ_y}{dt} + \frac{Q_y}{C} \right] = iV_0 \sin \omega t \\
i = V - I
\end{array} \right.
\]

where the 2nd equation we made. The trick is to solve for \( Q_x + iQ_y \) and then only keep the real part of the solution \( Q_x + iQ_y = Q \in \mathbb{R} + \text{id}t \) if the driving voltage is writing as \( V = V_0 \cos \omega t \) or the imaginary part if the driving voltage is \( \sin \omega t \).
\[
\frac{1}{L} \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = V_0 e^{i\omega t} e^{i\omega t} = e^{i\omega t} + e^{i\omega t} = e^{i\omega t} + e^{i\omega t} = e^{i\omega t} + e^{i\omega t}
\]

Try a solution of the form \(q = Q_0 e^{i\omega t}\). Steady state pol.

\[(-Lw^2 + iRw + 1/C)Q_0 e^{i\omega t} = V_0 e^{i\omega t}\]

This is true for all times if \(w = w\). \(Q_0 = \frac{V_0}{i\{Rw + i\{Lw^2 + 1/C\}\}}\)

And the current \(I(t) = \frac{d\phi}{dt} = i\omega V_0 e^{i\omega t} / i\{Rw + i\{Lw^2 + 1/C\}\}\)

\[I(t) = \frac{V_0 e^{i\omega t}}{R + i\{wL - 1/wc\}}\]

We remember from freshman physics where we deal with only resistive elements in the circuit then \(I = V/R\). Note we notice \(I = \frac{V}{\Sigma}\). We use a new name "Impedance". The current is given by \(I = \frac{V}{\Sigma}\) (impedance). Resistive impedance \(X_R = R\), Inductive Impedance \(X_L = i\omega L\), Capacitive impedance \(X_C = -i/wc\)

\[I = \frac{V}{X_R + X_L + X_C} = \frac{V_0 e^{i\omega t}}{R + i\{wL - 1/wc\}}\]

To calculate we remember that a complex quantity \(\alpha + i\beta = \sqrt{\alpha^2 + \beta^2} e^{i\theta}\) tangent \(\theta = \beta/\alpha\)

\[I = \frac{V_0 e^{i\omega t}}{R^2 + (wL - 1/wc)^2} \frac{1}{\sqrt{R^2 + (wL - 1/wc)^2}} e^{i\theta} = V_0 e^{i(\omega t - \theta)} / \sqrt{R^2 + (wL - 1/wc)^2}\]
and the real part is

\[ I = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} = \frac{V_0 \cos(\omega t - \phi)}{Z} \]

where \( Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2} \) = impedance
\( L\omega - 1/\omega C \) = Reactance
\( \tan \delta = \frac{L\omega - 1/\omega C}{R} \)

\[ Q = \int I dt = \frac{V_0}{Z} \omega \sin(\omega t - \phi) \]

\[ V_c = \frac{Q}{Z} = \left( \frac{V_0}{Z} \omega C \right) \sin(\omega t - \phi) \]

\[ V_L = L \frac{dI}{dt} = -\left( \frac{V_0}{Z} \omega \right) \sin(\omega t - \phi) \]

\[ V_R = IR = \left( \frac{V_0 R}{Z} \right) \cos(\omega t - \phi) \]

If we use the complex notation

\[ I = \frac{V_0 e^{i(\omega t - \phi)}}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \]

\[ V_L = L \frac{dI}{dt} = \frac{V_0 L \omega e^{i(\omega t - \phi)}}{Z} = \frac{V_0 L \omega e^{i(\omega t - \phi + \pi/2)}}{Z} \]

\[ V_c = X_C I = -\frac{V_0}{X_C} I X_C = -\frac{1}{\omega C} \left( \frac{V_0 e^{i(\omega t - \phi)}}{Z} \right) \]

\[ = \left( \frac{V_0}{Z} \omega C \right) e^{i(\omega t - \phi - \pi/2)} \]
We note that the magnitude of the current depends on the frequency of the driving voltage. This is like the spring being driven by a harmonic force. Hence you have the possibility of observing resonance behavior.

In a circuit, resonance is defined as that frequency when the current reaches a maximum. You can notice

\[ I = I_{\text{max}} \text{ when } Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \text{ is a minimum} \]

or when \( \omega = \frac{1}{\sqrt{LC}} \), the natural frequency of the circuit.

Note that at resonance \( \tan \delta = (\omega L - \frac{1}{\omega C})R = 0 \) or \( \delta = 0 \).

\[ I_{\text{max}} = \frac{V_0}{R} \cos \omega t = \frac{V_0}{R} \]

\[ Q = V_0 \omega \sin \omega t = \left( \frac{V_0}{R} \right) \sqrt{\frac{1}{L} \frac{1}{C}} \sin \omega t \]

\[ V_c = \frac{Q}{C} = \left( \frac{V_0}{R} \right) \sqrt{\frac{1}{L} \frac{1}{C}} \sin \omega t = \frac{V_0}{R} \sqrt{\frac{1}{L} \frac{1}{C}} \cos (\omega t - \frac{\pi}{2}) \]

\[ V_L = -L \left( \frac{V_0}{R} \right) \omega \sin \omega t = -\left( \frac{V_0}{R} \right) \sqrt{\frac{1}{L} \frac{1}{C}} \sin \omega t = \left( \frac{V_0}{R} \right) \sqrt{\frac{1}{L} \frac{1}{C}} \cos (\omega t + \frac{\pi}{2}) \]

This is just like Ohm's law, instead of using \( R \) you use the impedance of every circuit element. Hence in general

\[ V_c = X_c I = -\frac{i}{\omega C} I_0 \cos (\omega t - \delta) \]

\[ = I_0 \frac{1}{\omega C} \cos (\omega t - \delta - \frac{\pi}{2}) \]

\[ V_L = X_L I = i \omega L I_0 \cos (\omega t - \delta) = \omega L I_0 \cos (\omega t - \delta - \frac{\pi}{2}) \]
In the complex plane, if $\delta > 0$ \( \tan \delta = \omega L - \frac{1}{\omega C} \)
then current lags voltage.

If $\delta < 0$ then current leads voltage.

The power delivered to the circuit is

\[ P = IV = V_0 I_0 \cos \omega t \cos (\omega t - \delta) \]

Hence the power can be negative over the period of time $\cos \omega t$ and $\cos (\omega t - \delta)$ have opposite signs. Hence it makes more sense to talk about the average power/cycle

\[
\bar{P} = V_0 I_0 \int_0^{2\pi/\omega} \frac{\cos \omega t (\cos \omega t \cos \delta + \sin \omega t \sin \delta)}{2\pi} dt
\]

\[
= V_0 I_0 \frac{\int_0^{2\pi/\omega} \cos \omega t (\cos \omega t \cos \delta + \sin \omega t \sin \delta) dt}{2\pi}
\]

\[
= V_0 I_0 \frac{\int_0^{2\pi/\omega} (\cos^2 \omega t \cos \delta + \cos \omega t \sin \omega t \sin \delta) dt}{2\pi}
\]

\[
= V_0 I_0 \frac{\int_0^{2\pi/\omega} \cos^2 \omega t \cos \delta dt}{2\pi}
\]

since $\int_0^{2\pi/\omega} \cos \omega t \sin \omega t dt = 0$
Using $\int_0^{2\pi/\omega} \cos^2 \omega t \, dt = \frac{1}{2} \int_0^{2\pi/\omega} \, dt = \frac{\pi}{\omega}$

$\bar{P} = \frac{V_0 I_0 \cos \delta}{2}$

$= \frac{V_0 I_0 R}{2 \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} = \frac{1}{2} I_0^2 R \text{ at resonance}$

Using $I = \frac{V_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \cos (\omega t - \delta)$

$= \frac{V_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \cos (\omega t - \delta) = I_0 \cos (\omega t - \delta)$

$\bar{P} = \bar{I}^2 R = \frac{V_0^2}{2} \left( \int_0^{2\pi/\omega} \cos^2 (\omega t - \delta) \, dt \right) = I_0^2 R \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \cos^2 (\omega t - \delta) \, dt$

$= \frac{I_0^2 R \pi}{2\omega}$

$= \frac{I_0^2 R}{2}$
We now go further by considering circuits where the inductors and capacitors are in parallel with a resistor in series.

We can write three equations:

1. \( R \frac{dI_1}{dt} + \frac{(I_1 - I_2)}{C} = V_0 \cos wt \Rightarrow V_0 e^{iwt} \)
2. \( L \frac{d^2 I_2}{dt^2} + (\frac{I_2 - I_1}{C}) = 0 \)
3. \( R \frac{dI_1}{dt} + L \frac{d^2 I_2}{dt^2} = V_0 \cos wt \Rightarrow V_0 e^{iwt} \)

Note that (3) is the sum of (1) and (2).

Take eq. (1) and differentiate twice:

\( LCR \frac{d^3 I_1}{dt^3} + \frac{1}{C} \left( \frac{d^2 I_1}{dt^2} - \frac{d^2 I_2}{dt^2} \right) = -V_0 \omega^2 \cos wt = -V_0 \omega^2 e^{iwt} \)

Multiply through by LC:

4. \( LRC \frac{d^3 I_1}{dt^3} + L \left( \frac{d^2 I_1}{dt^2} - \frac{d^2 I_2}{dt^2} \right) = -V_0 LC \omega^2 \cos wt = -V_0 LC \omega^2 e^{iwt} \)

Add 4 and 3 and you get an equation for \( I_1 \):

\( LRC \frac{d^3 I_1}{dt^3} + L \frac{d^2 I_1}{dt^2} + R \frac{dI_1}{dt} = V_0 (1 - LC \omega^2) \cos wt \)

\( + V_0 (1 - LC \omega^2) e^{iwt} \)

where \( I_1 = I_{1x} + i I_{1y} \) when \( I \) uses \( e^{iwt} \)
In terms of current this becomes

\[ LRC \frac{d^2 I}{dt^2} + L \frac{dI}{dt} + RI = V_0 (1 - LCw^2) \cos wt \rightarrow V_0 (1 - LCw^2) e^{iwt} \]

where \( I = I_x + iI_y \). We try a steady state solution \( I_x = A e^{iwt} \). To get \( A \), we put this solution back in the diff.

\[ (LRCw^2 + iLw + R) A e^{iwt} = V_0 (1 - LCw^2) e^{iwt} \]

Clearly for this to be a solution for all times \( w' = w \). Then

\[ A = \frac{V_0 (1 - LCw^2)}{R - LRCw^2 + iLw} \]

Hence

\[ I = \frac{V_0}{\left( \frac{R + i \frac{Lw}{1 - LCw^2}}{iLCw} \right)} e^{iwt} = \frac{V_0 e^{i(\omega t - S)}}{Z} \]

\[ Z = \sqrt{R^2 + (Lw)^2} \]

\[ \tan S = \frac{Lw}{R (1 - LCw^2)} \]

Now we ask the interesting question whether we could get the same answer by writing

\[ I = \frac{V_0}{\sum X_i} \cdot \frac{V_0}{X_R + X_{\text{effective of the } L + C \text{ in parallel}}} \]
To do this we must ask how do impedances in parallel add. We know from the previous problem that impedances in series add like resistors in series.

\[ X_{\text{eff}} = X_R + X_L + X_C \quad X_R = R \quad X_L = i\omega L \quad X_C = -\frac{1}{i\omega C} \]

Note that \( X_C \) in series \( \rightarrow \frac{-1}{i\omega C} \rightarrow -\frac{1}{i\omega C} - i\frac{1}{\omega C} \)

Therefore \( \frac{1}{X_C} = \frac{1}{C_1} + \frac{1}{C_2} \) for capacitors in series as we know.

Hence we ask whether impedances in parallel add like resistors in parallel. So we try

\[
\frac{1}{X_{\text{eff}}} = \frac{1}{X_L} + \frac{1}{X_C}
\]

\[
= \frac{1}{i\omega L} + \frac{1}{-\frac{1}{i\omega C}} = \frac{1}{i\omega L} - \frac{\omega C}{i\omega L} = \frac{1}{i\omega L} + i\omega C
\]

\[
\frac{1}{X_{\text{eff}}} = \frac{1}{i\omega L} - \frac{\omega^2 LC}{i\omega L}
\]

\[
X_{\text{eff}} = \frac{i\omega L}{1 - \omega^2 LC}
\]

\[
Z = \sqrt{R^2 + (\omega L)^2}
\]

\[
\tan \delta = \frac{\omega L}{R(1 - \omega^2 LC)}
\]

\[
V = \frac{V_0 e^{i(\omega t - \delta)}}{Z}
\]

which is the same answer we got from solving the differential equation. All this is a big discovery.
So from here on we do not need to solve differential
equations when we deal with impedances. We have
impedances in series
\[ X_{\text{tot}} = \sum X_i \quad \text{where} \quad X_R = R \quad X_L = i\omega L \quad X_C = -\frac{1}{i\omega C} \]
impedances in parallel
\[ X_{\text{tot}} = \sum \frac{1}{X_i} \]
We have \[ I_1 = \frac{V_0}{Z} \cos(\omega t - \delta) \]
where \[ Z = \sqrt{R^2 + \left( \frac{V_0}{X_L} \right)^2} \quad \tan \delta = \frac{\omega R}{X_L(1 - \omega^2 C)} \]
ote that when \[ \omega = \sqrt{\frac{1}{LC}} \quad Z = \infty \quad \tan \delta = \infty \quad \delta = 90^\circ \quad I_1 = 0 \]
This is called an anti-resonant circuit \( \to I_1 = 0 \) at resonance
Now we solve for \( I_2 \). We use eq. (3)
\[
R I_1 + L \frac{dI_2}{dt} = V_0 \cos \omega t
\]
\[
L \frac{dI_2}{dt} = \frac{V_0}{Z} \left\{ \cos \omega t - \frac{R}{Z} \cos(\omega t - \delta) \right\}
\]
\[
I_2 = \frac{V_0}{L} \left\{ \sin \omega t - \frac{R}{Z} \sin(\omega t - \delta) \right\}
\]
Note that \( I_2 \) has the same frequency dependence as the source \( V \)
We solve the same \( I_2 \) using the impedance equations
\[
I_1 R + I_2 X_L = V_0 e^{i\omega t}
\]
\[
I_2 = \frac{1}{i\omega L} \left\{ V_0 e^{i\omega t} - R \frac{V_0}{Z} e^{i(\omega t - \delta)} \right\} \]
\[ I_2 = \frac{V_0}{Lw} \left\{ \frac{i(\omega t-\pi/2)}{Z} - \frac{Re^{i(\omega t-\pi/2)}}{Z} \right\} \]
\[ = \frac{V_0}{Lw} \left\{ \frac{\sin(\omega t) - R \sin(\omega t - \delta)}{Z} \right\} \]

Another example

\[ I_1 = \frac{V}{R_1 + X_{eq}} \]
\[ \frac{1}{X_{eq}} = \frac{1}{X_C} + \frac{1}{X_L + R_2 - \frac{i\omega C}{R_2 + i\omega L}} \]
\[ = i\omega C + \frac{1}{R_2 + i\omega L} = \frac{i\omega CR_2 - \omega^2 LC + 1}{R_2 + i\omega L} \]
\[ X_{eq} = \frac{(R_2 + i\omega L)(1 - \omega^2 LC + i\omega CR_2)}{(1 - \omega^2 LC)^2 + (R_2\omega C)^2} \]
\[ = \frac{R_2 - R_2^2\omega^2 LC + R_2\omega^2 LC + \omega L(1 - \omega^2 LC) - R_2^2\omega C}{(1 - \omega^2 LC)^2 + (R_2\omega C)^2} \]
\[ I_1 = \frac{V_0 e^{i\omega t}}{R_1 + \frac{R_2 + i\omega L(1 - \omega^2 LC - R_2\omega C)}{(1 - \omega^2 LC)^2 + (R_2\omega C)^2}} \]
Coupled Circuits in Forced Oscillations

\[ \begin{align*}
L_1 \frac{d^2q_1}{dt^2} + R_1 \frac{dq_1}{dt} + \frac{q_1}{C} + M \frac{d^2q_2}{dt^2} &= V_0 \cos \omega t = V_0 e^{i\omega t} \\
L_1 \frac{d^2q_1}{dt^2} + R_1 \frac{dq_1}{dt} + \frac{q_1}{C} &= V_0 e^{i\omega t} - M \frac{d^2q_2}{dt^2} \\
L_2 \frac{d^2q_2}{dt^2} + R_2 \frac{dq_2}{dt} &= -M \frac{d^2q_1}{dt^2}.
\end{align*} \]

Try solutions of the form \( q_1 = q_{10} e^{i\omega t}; q_2 = q_{20} e^{i\omega t} \). Again one determines that \( \omega' = \omega'' = \omega \). Plugging back you get

\[ (-L_1 \omega^2 + i R_1 \omega + \frac{1}{C_1}) q_{10} = V_0 + \omega^2 M q_{20} \]
\[ (-L_2 \omega^2 + i R_2 \omega + \frac{1}{C_2}) q_{20} = \omega^2 M q_{10} \]

or

\[ (-L_1 \omega^2 + i R_1 \omega + \frac{1}{C_1}) q_{10} = V_0 + \omega^2 M \left\{ \frac{\omega^2 M}{-L_2 \omega^2 + i R_2 \omega + \frac{1}{C_2}} \right\} q_{20} \]

or

\[ q_{1} = \frac{V_0 e^{i\omega t}}{\left\{ -L_1 \omega^2 + \frac{1}{C_1} + i R_1 \omega - \frac{M^2 \omega^4}{-L_2 \omega^2 + \frac{1}{C_2} + i R_2 \omega} \right\} } \]

Using \( q_1 = \frac{1}{i\omega} I_2 \),
\[ I_1 = \frac{V_0 e^{i \omega t}}{R_1 + j(\omega L_1 - \frac{1}{\omega C_1}) + \frac{M^2 \omega^2}{R_2 + j(\omega L_2 - \frac{1}{\omega C_2})}} \]

which is

\[ I_1 = \frac{V_0 e^{i \omega t}}{X_{R_1} + X_{L_1} + X_{C_1} + \frac{M^2 \omega^2}{X_{R_2} + X_{L_2} + X_{C_2}}} \]

We can show

\[ I_2 = \frac{V}{Z_1 + M^2 \omega^2 / Z_2} \quad I_2 = -i \omega M \quad \frac{V_0}{Z_2 / Z_1 + \omega^2 M^2 / Z_2} \]

\[ Z_1 = X_{R_1} + X_{L_1} + X_{C_1} \quad Z_2 = X_{R_2} + X_{L_2} + X_{C_2} \]

The analysis to determine the frequency dependence of the circuit gets quite complex.
Now we want to study what happens when you have repetitive elements in your circuit. For example,

\[ \frac{1}{2}Z \]
\[ \frac{1}{2}Z_1 \]
\[ \frac{1}{2}Z_2 \]
\[ \cdots \]
\[ \frac{1}{2}Z_{2n} \]
\[ \frac{1}{2}Z \]

Notice that in this case you are repeating the T-like configuration.

\[ \frac{1}{2}Z_1 \]
\[ \frac{1}{2}Z_2 \]
\[ Z_2 \]

The only difference is that at one end we add \( Z_0 \), the impedance of the voltage source, and at the other end we add the load impedance, \( Z_n \), of the circuit we are driving with the voltage generator.

There are cases where you want to consider your repetitive elements to be T-like configurations.

\[ \frac{1}{2}Z_1 + \frac{1}{2}Z_2 = Z_1 \]
\[ \frac{1}{Z_2} = \frac{1}{2}Z_2 + \frac{1}{2}Z_2 = \frac{1}{Z_2} \]
Notice the two arrays are identical at the middle and only they are different at the very beginning and the very end. It turns out that the only difference is the amount of current being transmitted to $Z_n$ but the frequency characteristic of the circuit (frequency characteristic of the current being transmitted).

Hence we will consider the $T$ configuration.

\[ \frac{1}{2} \frac{Z_2}{Z_1} \]

Let $V = V_0 e^{i\omega t}$ be the complex output voltage. We only keep the real part (Cos $\omega t$).

$Z$ = complex impedance. Namely $Z_1$ could be any combination of $R, L, C$ in series or in parallel. Similarly for $Z_2$. We will have the freedom to decide what we want.

\[
\begin{align*}
Z_0 i_0 + \frac{1}{2} Z_1 i_0 + Z_2 (i_0 - i_1) &= V \\
Z_2 (i_1 - i_0) + \frac{1}{2} Z_1 i_1 + Z_2 (i_1 - i_2) &= 0 \\
& \vdots \\
Z_2 (i_{n-1} - i_n) + \frac{1}{2} Z_1 i_{n-1} + Z_2 (i_{n-1} - i_n) &= 0 \\
Z_2 (i_{n-2} - i_n) + \frac{1}{2} Z_1 i_{n-2} + Z_2 (i_{n-2} - i_n) &= 0
\end{align*}
\]

We are interested in the steady state solution. We now know from our previous problems that the current will also have a time dependence $e^{i\omega t}$. 
Hence \( \ln = \ln_0 e^{\text{int} \tau} \)

where \( \ln_0 \) is a constant determined for the \( n \)th loop. We plug this solution into our series of equations

\[
\begin{align*}
Z_0 \ln_0 + \frac{1}{2} Z_1 \ln_0 - Z_2 (\ln_0 - \ln_1) - V_0 &= 0 \\
- Z_2 \ln_0 + (Z_1 + 2Z_2) \ln_0 - Z_2 \ln_0 &= 0 \\
&\vdots \\
- Z_2 \ln_{(n-2)} + (Z_1 + 2Z_2) \ln_{(n-1)} - Z_2 \ln_0 &= 0 \\
- Z_2 \ln_{n-1} + (Z_1 + 2Z_2) \ln_0 + Z_n \ln_0 &= 0
\end{align*}
\]

We rewrite the first and last relation to make it more symmetrical like the others. We write the first one as

\[-Z_2 \ln_{-1} + (Z_1 + 2Z_2) \ln_0 - Z_2 \ln_{10} = 0\]

where

\[-Z_2 \ln_{-1} = Z_0 \ln_0 - \frac{1}{2} Z_1 \ln_0 - Z_2 \ln_0 - V_0 \text{ or} \]

\[-\ln_{-1} = \left\{ \frac{(Z_0 - \frac{1}{2} Z_1 - Z_2) \ln_0 - V_0}{Z_2} \right\} \]

which represents

\[-Z_0 \ln_0 - \frac{1}{2} Z_1 \ln_0 - Z_2 \ln_0 - V_0 + (Z_1 + 2Z_2) \ln_0 - Z_2 \ln_{10} = 0\]

or

\[-Z_0 \ln_0 + \frac{1}{2} Z_1 \ln_0 + Z_2 (\ln_0 - \ln_{10}) - V_0 = 0\]

\[-Z_0 \ln_0 + \frac{1}{2} Z_1 \ln_0 - Z_2 (\ln_{10} - \ln_0) - V_0 = 0\]

which is the first loop equation.

Similarly, we write the last loop in the form

\[-Z_2 \ln_{(n-1)} + (Z_1 + 2Z_2) \ln_{(n-1)} - Z_2 \ln_{(n+1)} = 0\]

where

\[-Z_2 \ln_{(n+1)} = -Z_2 \ln_0 - \frac{1}{2} Z_1 \ln_0 + Z_n \ln_0\]