PHYSICS 3320

February 7, 2013

Homework 2 Solution

1) Consider a long thin conducting cylindrical shell of radius “a” with its axis of symmetry in the z direction. The cylinder is cut in half along a direction that we will call the x-axis. The two halves are separated by a such a small distance that you can neglect it. The upper half of the cylinder is raised to a potential +Φ₀; the lower half is lowered to a potential -Φ₀. Obtain an expression for the potential in terms of ρ and φ everywhere outside the cylinder (ρ > a). Leave the answer as an infinite sum.

SOLUTION: The most general solution for the potential for ρ > a is:

\[ \Phi(\rho \geq a, \phi) = \sum_{n=1}^{\infty} \rho^{-n} \{A_n \cos(n\phi) + B_n \sin(n\phi)\} \]

To solve for the constant we apply the stated boundary condition at ρ = a, namely the value of the potential. We recognize we have a Fourier series once ρ is set equal to a. Hence

\[ \int_{0}^{\pi} \Phi_0 \cos(m\phi) d\phi - \int_{\pi}^{2\pi} \Phi_0 \cos(m\phi) d\phi = \]

\[ \sum_{n=1}^{\infty} a^{-n} \{A_n \int_{0}^{2\pi} \cos(m\phi) \cos(n\phi) d\phi + B_n \int_{0}^{2\pi} \cos(m\phi) \sin(n\phi) d\phi\} \]

The integral on the left side is zero. On the right all the integrals with \( \cos(m\phi)\sin(n\phi) \) are zero. The only term that is non zero is the integral with \( \cos(m\phi)\cos(n\phi) \) when \( n = m \). We discussed this last semester when we discussed Fourier series. The value of the integral is \( \pi \). Hence we get:

\[ \pi a^{-m} A_m = 0 \]

Hence all the \( A_n \) terms are zero.

To get the \( B_n \) coefficients we write:

\[ \int_{0}^{\pi} \Phi_0 \sin(m\phi) d\phi - \int_{\pi}^{2\pi} \Phi_0 \sin(m\phi) d\phi = \]
\[
\sum_{n=1}^{\infty} a^{-n} \{ A_n \int_0^{2\pi} \sin(n\phi) \cos(n\phi) d\phi + B_n \int_0^{2\pi} \sin(n\phi) \sin(n\phi) d\phi \}
\]

This leads to the equation:

\[
\pi a^{-m} B_m = -\frac{1}{m} \Phi_0 \{ \cos(m\phi) \big|_0^\pi - \cos(m\phi) \big|_\pi^{2\pi} \} = 0
\]

\[
\pi a^{-m} B_m = 0 \quad m = \text{even} = 0, 2, 4, \ldots
\]

\[
\pi a^{-m} B_m = \frac{4}{m} \Phi_0 \quad m = \text{odd} = 1, 3, 5, \ldots
\]

Hence we arrive at the answer:

\[
\Phi(\rho \geq a, \phi) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{n} a^n \rho^{-n} \sin(n\phi)
\]

2) Show that the equation written in terms of the relativistic 4 dimensional notation:

\[
\partial_\mu F_{\mu\nu} = -\frac{4\pi}{c} j_\nu
\]

where

\[
J_\nu = (j_x, j_y, j_z, ic\rho)
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad A_\mu = (A_x, A_y, A_z, ic\phi)
\]

\[
\partial_\mu = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{ic} \frac{\partial}{\partial t} \right)
\]

leads to the two Maxwell equations:

\[
\vec{\nabla} \cdot \vec{E} = 4\pi \rho
\]

\[
\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}
\]

SOLUTION: Remember, repeated indicies mean you sum over them. Hence we sum over \( \mu = 1 - 4 \). Choose \( \nu = 1 \). We get:

\[
\frac{\partial F_{11}}{\partial x} + \frac{\partial F_{21}}{\partial y} + \frac{\partial F_{31}}{\partial z} + \frac{1}{ic} \frac{\partial F_{41}}{\partial t} = -\frac{4\pi}{c} j_1
\]
Putting in the components of \( B \) and \( E \) described by the \( F \)'s as done in last week's homework we get:

\[
\frac{\partial 0}{\partial x} + \frac{\partial (-B_z)}{\partial y} + \frac{\partial B_y}{\partial z} + \frac{1}{ic} \frac{\partial E_z}{\partial t} = -\frac{4\pi}{c} j_z \text{ or } \\
(\vec{\nabla} \times \vec{B})_x = \frac{4\pi}{c} j_z + \frac{1}{c} \frac{\partial E_x}{\partial t}
\]

This is the \( x \) component of the second Maxwell equation above. By choosing \( \nu = 2 \text{ and } 3 \) we get the \( y \) and \( z \) components of the same equation. Then for \( \nu = 4 \) we get:

\[
\frac{\partial F_{14}}{\partial x} + \frac{\partial F_{24}}{\partial y} + \frac{\partial F_{34}}{\partial z} + \frac{1}{ic} \frac{\partial F_{44}}{\partial t} = -\frac{4\pi}{c} j_4 = -4\pi \rho \text{ or } \\
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} + \frac{1}{ic} \frac{\partial 0}{\partial t} = 4\pi \rho \text{ or } \\
\vec{\nabla} \cdot \vec{E} = 4\pi \rho
\]

which is the first Maxwell equation equation above.

3) Consider a charged particle of mass “m” and charge “e” moving in a plane perpendicular to the direction of a constant magnetic field \( B \). Show that the particle follows a circular orbit. Let the direction of \( B \) define the \( z \)-direction.

HINT: Use Cartesian coordinates. Choose a coordinate system so that \( \dot{y} = dy/dt = 0 \) when \( x = 0 \). Solve the set of coupled differential equation resulting from \( F = ma \). Write your solution in terms of:

\[
\omega_0 = \frac{eB}{mc}
\]

where \( f_0 = (1/2\pi)\omega_0 \) is known as the cyclotron frequency. This is the basic parameter of accelerators.

SOLUTION: Given that \( B \) defines the \( z \)-direction the equation of motion is:

\[
\vec{F} = m\left\{ \frac{\partial^2 x}{\partial t^2} \hat{i} + \frac{\partial^2 y}{\partial t^2} \hat{j} + \frac{\partial^2 z}{\partial t^2} \hat{k} \right\} =
\]
\[
q \frac{j}{c} \times \vec{B} = q \frac{1}{c} B \{ \frac{\partial y}{\partial t} i - \frac{\partial x}{\partial t} j \}
\]

This leads to three equations, one for each component as follows:

\[
\frac{\partial^2 x}{\partial t^2} = \frac{qB}{mc} \frac{\partial y}{\partial t} = w_o \frac{\partial y}{\partial t} \quad eqn \ (1)
\]

\[
\frac{\partial^2 y}{\partial t^2} = -\frac{qB}{mc} \frac{\partial x}{\partial t} = -w_o \frac{\partial x}{\partial t} \quad eqn \ (2)
\]

\[
\frac{\partial^2 z}{\partial t^2} = 0 \quad eqn \ (3)
\]

Equation (3) leads to \( \frac{\partial z}{\partial t} = C \) where C is a constant. Given the initial condition that \( v_x = 0 \) at \( t = 0 \) the constant is 0. Hence \( z = z_0 \).

Equation (2) leads to:

\[
\frac{\partial y}{\partial t} = -w_0 x + C \quad eqn \ (4)
\]

Given the initial condition that \( \frac{\partial y}{\partial t} = 0 \) when \( x = 0 \) we have \( C = 0 \). Substituting this result into equation (1) we get:

\[
\frac{\partial^2 x}{\partial t^2} = -w_0^2 x
\]

The solution to this differential equation is:

\[
x = a_0 \cos(w_0 t) + b_0 \sin(w_0 t) = A_0 \cos(w_0 t + \delta) \quad eqn \ (5)
\]

where \( A_0 \) and \( \delta \) are constants. Taking \( \frac{\partial x}{\partial t} \) from equation (5) and substituting into equation (2) we get:

\[
\frac{\partial^2 y}{\partial t^2} = -w_0^2 A_0 \sin(w_0 t + \delta) \quad eqn \ (6)
\]

\[
y = A_0 \sin(w_0 t + \delta) \quad eqn \ (7)
\]

This leads to the conclusion that \( x^2 + y^2 = A_0^2 \) a constant independent of time. This is the equation of a circle. Since in this case \( z \) is a constant it remains a circle. If \( z \) had an initial velocity then we get a helix.

(b) Solve the same problem in cylindrical coordinates with the initial conditions at \( t = 0 \) \( \dot{r} = 0 \), \( \ddot{r} = 0 \) then show \( r = r_0 \) (constant) at all times.
This is an interesting problem. Try to find constants of the motion. The condition at \( t = 0 \) \( \dot{\rho} = 0 \) implies that the initial direction of the motion is well defined. State it?

**SOLUTION:** The equations of motion in cylindrical coordinates is given by:

\[
\vec{F} = q \frac{\dot{\phi}}{c} \times \vec{B} = q \frac{\dot{\rho}}{c} \left| \begin{array}{ccc}
\dot{\rho} & \hat{\phi} & \dot{z} \\
\dot{\phi} & \rho \dot{\phi} & \hat{z} \\
0 & 0 & B
\end{array} \right|
\]

This leads to the two (non-z-component) equations of motion:

\[
F_\rho = m \{ \ddot{\rho} - \rho \dot{\phi}^2 \} = \frac{qB}{c} \rho \dot{\phi} \quad eqn \ (1)
\]

\[
F_\phi = m \{ 2 \dot{\rho} \dot{\phi} + \rho \ddot{\phi} \} = - \frac{qB}{c} \dot{\rho} \quad eqn \ (2)
\]

We want to show that \( \rho = \rho(t=0) \) namely it is constant. This is the requirement for circular motion. Equation (2) can be written in the form:

\[
\frac{m}{\rho} \frac{\partial}{\partial t} (\rho^2 \dot{\phi}) = - \frac{qB}{c} \dot{\rho} \quad eqn \ (3)
\]

This leads to the following series of equations:

\[
\frac{\partial}{\partial t} (\rho^2 \dot{\phi}) = - \frac{w_0 \partial \rho^2}{2 \partial t}
\]

\[
\frac{\partial}{\partial t} \{ \rho^2 \dot{\phi} + \frac{w_0}{2} \rho^2 \} = 0 \quad eqn \ (4)
\]

\[
\rho^2 \dot{\phi} + \frac{w_0}{2} \rho^2 = C \quad eqn \ (5)
\]

This shows a constant of the motion which, at least I, can not interpret as a result of some fundamental physical principle like angular momentum or energy conservation. Using the initial conditions (t=0) we have \( \dot{\phi} = -w_0 \) at \( t=0 \). This indicates that at \( t=0 \) \( F_\rho \) is in the \( \dot{\rho} \) direction. Hence we get for C:

\[
C = -\rho_0^2 w_0 + \frac{w_0}{2} \rho_0^2 = -\frac{w_0}{2} \rho_0^2
\]

Putting this value of C into equation (5) we get:
\[ \rho^2 \dot{\phi} = -\frac{w_0}{2} (\rho^2 + \rho_0^2) \text{ or} \]
\[ \dot{\phi} = -\frac{w_0}{2} \left\{ \frac{\rho_0^2}{\rho_2} + 1 \right\} \text{ eqn (6)} \]

Putting equation (6) into equation (1) we get:

\[ \ddot{\rho} = \rho \dot{\phi} (w_0 + \dot{\phi}) = \rho \frac{w_0}{2} \left( \frac{\rho_0^2}{\rho^2} + 1 \right) \left\{ w_0 - \frac{w_0}{2} \left( \frac{\rho_0^2}{\rho^2} + 1 \right) \right\} \text{ or} \]
\[ \ddot{\rho} = -\rho \frac{w_0}{2} \left\{ \frac{\rho_0^2}{\rho^2} + 1 \right\} \left\{ \frac{w_0}{2} \left( 1 - \frac{\rho_0^2}{\rho^4} \right) \right\} \text{ or} \]
\[ \dot{\rho} = -\rho \frac{w_0}{4} \left( 1 - \frac{\rho_0^2}{\rho^4} \right) \text{ or} \]
\[ \ddot{\rho} = \frac{w_0^2}{4 \rho^2} (\rho_0^4 - \rho^4) \text{ eqn (7)} \]

We now can form another constant of the motion by writing:

\[ \frac{\partial}{\partial t} \rho^2 = 2 \dot{\rho} \ddot{\rho} = \frac{w_0^2}{2 \rho^3} (\rho_0^4 - \rho^4) \dot{\rho} \text{ or} \]
\[ \frac{\partial}{\partial t} \rho^2 = \frac{w_0^2}{2} \left( \frac{\rho_0^4}{\rho^3} - \rho \right) \dot{\rho} \text{ or} \]
\[ \frac{\partial}{\partial t} \rho^2 = -\frac{w_0^2}{4} \frac{\partial}{\partial t} \left( \frac{\rho_0^4}{\rho^2} + \rho^2 \right) \]

Putting all derivatives on the same side we get a constant of the motion equation:

\[ \frac{\partial}{\partial t} \left\{ \rho^2 + \frac{w_0^2}{4} \left( \frac{\rho_0^4}{\rho^2} + \rho^4 \right) \right\} \text{ or} \]
\[ \left\{ \dot{\rho}^2 + \frac{w_0^2}{4} \left( \frac{\rho_0^4}{\rho^2} + \rho^4 \right) \right\} = D \text{ eqn (8)} \]

where D is a constant which is determined by initial conditions and becomes:
\[ D = \frac{w_0^2}{2} \rho_0^2 \]
Putting this value of $D$ into the equation (8) and solving for $\rho^2$ we get:

$$\rho^2 = \frac{u_0^2}{2} \left\{ \rho_0^2 - \frac{1}{2} \left( \frac{\rho_0^4 + \rho^4}{\rho^2} \right) \right\}$$

After simplifying this we get the final equation:

$$\rho^2 = - \frac{u_0^2}{4\rho} \left\{ \rho_0^2 - \rho^2 \right\}^2 \text{ eqn (9)}$$

Since both sides are quadratics and they are negative of one another the only consistent solution is $\dot{\rho} = 0$ and $\rho = \rho_0$ at all times. Hence the particle moves in a circle with the same radius as the initial radius, which is the solution we wanted to determine.

(4) A very long rectangular conductor of height “a” and width “b” carries a current “I” in the direction shown. A magnetic field “B” is applied in the direction of the width as shown. Let the number of electrons/unit volume (cm$^3$) be $N$.

Calculate the magnitude and sign of the potential difference across “a” if the particles that move are electrons (in the direction opposite to the current direction) and if the particles that move are protons (in the direction of the current). This is known as the “Hall Effect” and showed that the particles that move are electrons.

SOLUTION: The force on both the positive charges moving in the direction of the current and the negative charges moving in the opposite direction is upwards. If the positive charges are the ones moving then the potential at the top is higher than the potential at the bottom. If the negative charges are the ones that move the potential at the top is lower than the potential at
the bottom. Hence the sign of the potential difference between the top and bottom of the bar tells you which charges are the ones moving.

The charges will continue to move due to the Lorentz force until a large enough Electric field builds up to cancel this force. The potential difference is equal to the Electric field times the distance between the two surfaces which is "a". Hence:

\[ q\vec{E} = -q\frac{\vec{v}}{c} \times \vec{B} \]

\[ \delta \Phi = Ea \]

\[ \delta \Phi = \frac{v}{c}Ba = \frac{1}{cNeab}Ba = \frac{BI}{cNeb} \]

(5) Consider an infinitely long solid cylindrical conductor or radius "R," carrying a constant current "I" in the direction of its axis.

(a) Using the Biot-Savart Law set up the integral for the magnetic field "B". Try to do the integral for \( r > R \) and \( r < R \) but do not spend too much time.

**SOLUTION:** The integral is:

\[ \vec{B} = \int \int \int \frac{j(r') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3r' \]

Using cylindrical variables in cartesian unit vectors we have the following:

\[ \vec{r} = \rho \cos(\phi)\hat{i} + \rho \sin(\phi)\hat{j} + z\hat{k} \]

\[ \vec{r}' = \rho' \cos(\phi')\hat{i} + \rho' \sin(\phi')\hat{j} + z'\hat{k} \]

We now use the symmetry of the problem and note that the answer must be independent of \( \phi \) and \( z \). Hence choose both of them = 0. This gives us:

\[ \vec{B} = \int \int \int j(r') \frac{\rho' \sin(\phi')\hat{i} + (\rho - \rho' \cos(\phi'))\hat{j}}{\{\rho^2 - 2\rho' \cos(\phi') + \rho'^2 + z'^2\}^{3/2}} \rho' d\rho' d\phi' dz' \]

We know from the Biot Savart law that if the current "I" or current density "\( j \)" is in the \( \hat{k} \) direction the magnetic field "B" is in circles in the x-y plane. Hence for our situation, where \( \phi = 0 \) the B field must be in the \( \hat{j} \) direction. Hence the integral of the \( \hat{i} \) component must be zero. This, in fact, is easy
to see, since the magnitude of the \( \hat{z} \) part of the function (being integrated over \( d\phi' \)) from 0 to \( \pi \) is exactly the opposite to the magnitude from \( \pi \) to 2\( \pi \). Hence we have that \( \vec{B} \) is in the \( \hat{j} \) direction at \( \phi = 0 \) or in general it is in the \( \hat{\phi} \) direction. The integral over \( z' \) is easy since

\[
\int_{-\infty}^{\infty} \frac{dz'}{(a^2 + z'^2)^{3/2}} = \frac{1}{a} \left[ \frac{z'}{a^2 + z'^2} \right]_{-\infty}^{\infty} = \frac{2}{a}
\]

Hence we have left:

\[
B = 2j \int \int \frac{\rho - \rho' \cos(\phi')}{\{4 \rho^2 + \rho^2 - 2\rho \rho' \cos(\phi')\}^{1/2}} \rho' d\rho' d\phi'
\]

This integral does not seem easy to do so go no further.

(b) Do the same for the vector potential \( "A" \). Try to do the integrals but do not spend too much time.

SOLUTION: The integral is:

\[
\vec{A} = \int \int \int \frac{\vec{j}}{(\rho^2 + \rho'^2 - 2\rho \rho' \cos(\phi') + z'^2)^{1/2}} \rho' d\rho' d\phi' dz' \vec{k}
\]

This integral is equally difficult as the one for the B field. Hence go no further.

(c) Do the same problem for \( r > R \) and \( r < R \) using Ampere’s Law. Explain in detail how you get the answer. This should show the usefulness of this Law.

SOLUTION: Apply Ampere’s Law; The integral over a closed loop \( \int \vec{B} \cdot d\vec{l} = 4\pi I \) where \( I \) is the current going through the loop. By the Biot-Savart Law we know the “B” field goes in circles with “I” going through the circle. Hence choose you integral path to be a circle of radius “r”. Then, by symmetry, \( \vec{B} = B(\rho) \) in the direction of the tangent of the circle. Hence for \( \rho > R \) the integral becomes:

\[
\int \vec{B} \cdot d\vec{l} = B2\pi \rho = 4\pi I
\]

\[
\vec{B} = \frac{2I}{r} \hat{\phi}
\]

9
For \( r < R \) the circle about which you are taking the integral has less current going through it. The current \( "I" \) becomes \( \{I/(\pi R^2)\} \pi r^2 \). Following the same argument as above we get:

\[
\vec{B} = \frac{2I}{r} \hat{\varphi} = \frac{2I}{R^2} r \hat{\varphi}
\]

6) A very long solid cylindrical conductor or radius "a" has a cylindrical hole of radius "b" (b < a). The axis of the conductor and of the hole are parallel and separated by a distance "c" as shown in the figure. The conductor carries a constant current I along its axis. Find the magnitude and direction of the magnetic field everywhere in the hole; show that it is a constant both in magnitude and direction.

**SOLUTION:** The resultant field is the superposition (sum) of 2 B fields. One (B) is due to the full cylinder without a hole with the current out of the plane of the paper and the other one (B') is due to the same current density in the small hole but in the opposite direction. Let \( \vec{r} \) be the vector from the center of the big cylinder to a location in the space where the small hole is. Let \( \vec{r}' \) be the vector from the center of the small hole to the same location. Note that the vector \( \vec{r} - \vec{r}' \) is the vector \( \vec{c} \) which is the vector from the center of the large cylinder to the center of the small hole. Hence we have using Ampere's Law:

\[
\vec{B} = \frac{2I}{a^2} r \hat{\varphi} = 2\pi j r (\hat{j} \times \hat{r}) = 2\pi j \vec{r} \times \vec{r}
\]

\[
\vec{B}' = -2\pi j \vec{r} \times \vec{r}'
\]

The minus sign in \( \vec{B}' \) is because the current is in the opposite direction so that we can mimic the empty space which has no net current. Hence the superposition (sum) of the two fields becomes:

\[
\vec{B} + \vec{B}' = 2\pi j \vec{r} \times (\vec{r} - \vec{r}') = 2\pi j \vec{c}
\]

Hence it is a constant. Note that \( j = I/\pi(a^2 - b^2) \) is the correct current density.

7) **SPECIAL PROBLEM.** This is the special credit problem. It is the basis for all present day accelerator designs. Suppose you have a circular tunnel with a constant magnetic field \( \vec{B} \) pointing down. Let that be the z-direction.
You send a beam in pointing in an arbitrary direction. Now at $t = 0$ you have $\rho = \rho_0$, $\dot{\rho} = \dot{\rho}_0$, $\dot{\phi} = 0$, $\dot{z} = \dot{z}_0$. You can choose your origin and orientation of coordinates so that at $t = 0$ we have $z = 0$ and $\phi = 0$.

(a) Write the equations of motion in cylindrical coordinates from $F = ma$ where $F$ is given by the Lorentz force. You will have 2 differential equations, one for each of the possible components of the Lorentz force (in cylindrical coordinates).

(b) Derive a constant of the motion using the differential equation from the force equation in the $\phi$ direction.

(c) Solve for $\dot{\phi}$ in terms of the cyclotron frequency, the variable $\rho$ and the initial value of $\rho$.

(d) Put this solution into the force equation in the $\dot{\rho}$ direction. Solve for $\ddot{\rho}$. Obtain another constant of the motion with the trick

$$\frac{\partial \rho^2}{\partial t} = 2\dot{\rho}\ddot{\rho}$$

(e) Solve for $\rho^2$.

(f) Obtain a solution for $\rho$ as a function of time.
Replacing into (1) we get
\[ \frac{d^2 x}{dt^2} = -w_0 x \]
\[ \therefore x = A_0 \cos(w_0 t) + b_0 \sin(w_0 t) = A_0 \cos(w_0 t + \delta) \]
so the solution is

Replacing \( \frac{dx}{dt} \) into (2) we get
\[ \frac{d^2 y}{dt^2} = -w_0 A_0 \sin(w_0 t + \delta) \]
The solution is
\[ y = A_0 \sin(w_0 t + \delta) \]

We note \( x^2 + y^2 = A_0^2 = \text{const.} \)

Hence the equation of the path is a circle. If you add the \( \mathbb{F} \) motion \( \mathbb{F} = \text{const.} \) then it remains a circle, if \( \mathbb{F} \) is moving initially we have a helix.

b) In cylindrical coordinates
\[ \mathbb{F} = \frac{q}{c} \mathbb{v} \times \mathbb{B} = \frac{q}{c} \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{1}{c} \end{vmatrix} = \frac{qB}{c} (\rho \hat{\phi} - \phi \hat{\rho}) \]

Writing the eq. for \( \mathbb{F} \) for each component

1. \[ F_\rho = m(\dot{\rho} - \rho \dot{\phi}^2) = \frac{qB}{c} \rho \dot{\phi} \]
2. \[ F_\phi = m(\dot{\phi}^2 + \rho \ddot{\phi}) = -\frac{qB}{c} \dot{\rho} \]
The 2nd equation can be written in the form

\[
\frac{m}{\rho} \frac{d}{dt} (\rho^2 \dot{\rho}) = -\frac{qB}{c} \rho
\]

We need to show \( \dot{\rho} = 0 \). From eq. 1, we choose initially \( \rho = \rho_0 \) such that \( \dot{\phi} = \omega \mathcal{S} = -\frac{qB}{mc} \) at \( t=0 \). Then \( \dot{\rho} = 0 \) at \( t=0 \). We also choose initially \( \dot{\rho} = 0 \).

Then \( \ddot{\rho} = 0 \) and \( \rho = \text{const} \) which means \( \dot{\rho} = 0 \) always since \( \rho = 0 \) at \( t=0 \).

Then we get

\[
\dot{\rho} = \frac{eB}{mc} \quad \rho = \text{const.}
\]

This is the eq. of a circle.

A more complicated procedure fails to show \( \rho = \rho(t=0) \) follows.

Take eq. 1

\[
\ddot{\rho} = \omega \rho \dot{\rho} + \rho \ddot{\rho} = \rho \dot{\rho} (\omega + \dot{\rho})
\]

From (3) we have

\[
\frac{d}{dt} (\rho^2 \dot{\rho}) = -\omega \rho \dot{\rho} = -\frac{\omega}{2} \frac{d}{dt} \rho^2
\]

\[
\frac{d}{dt} (\rho^2 \dot{\rho} + \frac{\omega}{2} \rho^2) = 0
\]

\[
\rho^2 \dot{\rho} + \frac{\omega}{2} \rho^2 = C
\]
Let $p' = 0 \quad \rho = \rho_0 \quad \dot{y} = -\omega \quad \text{at} \quad t = 0$

\[-p^2 \omega + \frac{\omega}{2} \rho_0^2 = C\]

\[C = -\frac{\omega}{2} \rho_0^2\]

Plugging back into (6) we get:

\[\rho^2 \dot{p} = -\frac{\omega}{2} \rho^2 - \frac{\omega}{2} \rho_0^2 = -\frac{\omega}{2} (\rho^2 + \rho_0^2)\]

\[\dot{y} = -\frac{\omega}{2} \left( \frac{\rho_0^2}{\rho^2} + 1 \right)\]

We insert this solution into (1):

\[\ddot{\rho} = \rho \dot{p} \omega' (\omega + \dot{y}) = -\frac{\rho \omega}{2} \left( \frac{\rho_0^2}{\rho^2} + 1 \right) \left[ \frac{\omega}{2} - \frac{\omega}{2} \left( \frac{\rho_0^2}{\rho^2} + 1 \right) \right]\]

\[= -\frac{\rho \omega}{2} \left( \frac{\rho_0^2}{\rho^2} + 1 \right) \left[ \frac{\omega}{2} \left( 1 - \frac{\rho_0^2}{\rho^2} \right) \right]\]

\[= -\frac{\rho \omega}{4} \left( 1 - \frac{\rho_0^4}{\rho^4} \right)^2\]

\[\dot{p} = \frac{\omega}{4\rho^3} \left( \rho_0^2 - \rho^2 \right)\]
Now consider
\[
\frac{d}{dt} \rho^2 + \frac{\omega^2}{2} (\rho_0^4 + \rho^4) = 0
\]

\[\therefore \rho^2 + \frac{\omega^2}{4} (\rho_0^4 + \rho^4) = C\]

Using our initial conditions
\[C = \frac{\omega^2 \rho_0^2}{2}\]

\[\therefore \rho^2 = C - \frac{\omega^2 (\rho_0^4 + \rho^4)}{4} = \frac{\omega^2}{2} \left\{ \frac{\rho_0^2}{2} - \left( \frac{\rho_0^4 + \rho^4}{\rho^2} \right) \right\} = \frac{\omega^2}{4 \rho^2} \left\{ 2 \rho_0^2 - \rho_0^4 - \rho^4 \right\} = -\frac{\omega^2}{4 \rho^2} \left\{ \rho^2 - \rho_0^2 \right\}^2\]

\[\rho^2 = -\frac{\omega^2}{4 \rho^2} \left\{ \rho^2 - \rho_0^2 \right\}^2\]
The only possible solution because of the - sign and both sides being quadratics is

\[ \rho^2 = 0 \]
\[ \rho^2 = \rho_0^2 \]

This leads then to the same conclusions.
What happens if the initial conditions are

\[ \dot{p} = 0 \quad p = p_0 \quad \text{but} \quad \dot{\phi} = \omega - \omega_0 \neq 0 \]

Then solution for \( \phi \) from eq. 4.10:

\[-p_0^2 \frac{\omega}{2} + \frac{\omega_0}{2} p_0^2 = C \]

\[ C = p_0^2 \left( \frac{\omega}{2} - \omega_0 \right) \]

\[ p^2 \dot{\phi} = C - \frac{\omega}{2} p^2 \]

\[ = p_0^2 \left( \frac{\omega}{2} - \omega_0 \right) - \frac{\omega}{2} p^2 \]

\[ \dot{\phi} = \frac{p_0^2}{p^2} \left( \frac{\omega}{2} - \omega_0 \right) - \frac{\omega}{2} \]

Value for \( \ddot{p} \):

\[ \ddot{p} = p \ddot{\phi} (\omega + \dot{\phi}) = \left[ \frac{p_0^2}{2} \left( \frac{\omega}{2} - \omega_0 \right) - \frac{\omega p}{2} \right] \left[ \frac{\omega + p_0^2 (\omega - \omega_0)}{2} \right] \]

Multiply by \( 2 \dot{p} \)
\[2 \ddot{p}^2 = 2\dot{p} \dot{p} \left\{ \frac{p_0^2}{p^2} (w - w_0) - \frac{w}{2} \right\} \left\{ \frac{p_0^2}{p^2} (w - w_0) + \frac{w}{2} \right\} = 2\dot{p} \dot{p} \left\{ \frac{p_0^4}{p^4} (w - w_0)^2 - \frac{w^2}{4} \right\} \]

\[\frac{d}{dt} \dot{p}^2 = \frac{d}{dt} \left\{ -\frac{p_0^4}{p^2} (w - w_0)^2 - \frac{w^2}{4} \dot{p}^2 \right\} \]

\[\therefore \dot{p}^2 + \frac{p_0^4}{p^2} (w - w_0)^2 + \frac{w^2}{4} \dot{p}^2 = C\]

Using the initial conditions we get

\[C = p_0^2 (w - w_0)^2 + \frac{w^2}{4} \left( \frac{p_0^2}{p^2} \right) = p_0^2 \left( \text{initial speed} \right)\]

Multiply by \(4\dot{p}^2\)

\[4\dot{p} \dot{p}^2 = 4p_0^2 (w - 2w_0)^2 + w^2 \dot{p}^2 - p_0^4 (w - 2w_0)^2 - w^2 \dot{p}^2 \]

Let \(u = \dot{p}^2\)

\[\ddot{u} = 2\dot{p} \dot{p}
\]

\[\ddot{u} = u \frac{p_0^2}{p^2} [(w - 2w_0)^2 + w^2] = w^2 u - p_0^4 (w - 2w_0)^2\]

This \(u\) of the form

\[u^2 = au^2 + bu + c\]

where

\[a = -w^2\]

\[c = -p_0^4 (w - 2w_0)^2\]

\[b = p_0^2 \left( \frac{w - 2w_0)^2 + w^2}{} \right)\]
So we get
\[ \frac{du}{\sqrt{a\nu^2 + b\nu + c}} = e^t \]

For \( a < 0 \) the integral gives
\[ \frac{1}{\sqrt{-a}} \text{asin}^{-1} \left\{ \frac{-2a\nu - b}{\sqrt{b^2 - 4ac}} \right\} \quad u = \rho^2 \]

Replacing the constants
\[ \frac{1}{\nu} \text{asin}^{-1} \left\{ \frac{2\omega^2 p^2 - \rho_0^2 \left[ (\omega - 2\omega_0)^2 + \omega^2 \right]}{\rho_0^4 \left[ (\omega - 2\omega_0)^2 + \omega^2 \right]^2 - 4\omega^2 \rho_0^4 (\omega - 2\omega_0)^2} \right\} = t + c \]

\[ 2\omega^2 \rho^2 - \rho_0^2 \left[ (\omega - 2\omega_0)^2 + \omega^2 \right] = \frac{2\sin(\nu + c)}{\rho_0^4 \left[ (\omega - 2\omega_0)^2 + \omega^2 \right]^2} \]

get \( \sqrt{-b} \)

\[ 2\omega^2 \rho^2 = \rho_0^2 \left[ (\omega - 2\omega_0)^2 + \omega^2 \right] + \rho_0^2 \left[ - (\omega - 2\omega_0)^2 + \omega^2 \right] \sin(\nu + c) \]

\[ \rho^2 = \rho_0^2 \left[ \frac{(\omega - 2\omega_0)^2}{2\omega^2} \left( 1 + \sin(\omega t) \right) + \frac{1}{2} \left( 1 + \sin(\omega t) \right) \right] \]

\[ = \rho_0^2 \left[ \frac{1}{2} \left( 1 + \sin(\omega t) \right) + \frac{1}{2} \left( 1 + \sin(\omega t) \right) \right] \frac{2\omega_0}{\omega} \left( 1 - \frac{\omega_0}{\omega} \right) \left( 1 + \sin(\omega t) \right) \]

\[ = \rho_0^2 \left[ \frac{1}{2} \cdot \frac{2\omega_0}{\omega} \left( 1 - \frac{\omega_0}{\omega} \right) \left( 1 + \sin(\omega t) \right) \right] \]

It must be corrected instead of \( \sin(\omega t) \) so that \( \rho = \rho_0 \) at \( t=0 \)
\[ \therefore \omega c = \frac{\pi}{2} \]