
Outline
- Examples of point charge radiation
  - Radiation Reaction

Two Examples: braking radiation and synchrotron radiation
see Oct 29 lecture notes

We will also cover homework-related examples later in this lecture.
First, let's get familiar with the concept of radiation reaction.

A charged particle accelerates (decelerates)

\[
\text{Total power radiated } P_{\text{rad}} = \frac{\mu_0 \varepsilon_0^2}{6\pi c} \quad \text{(for Vccc)}
\]

\[
\text{Energy is conserved, so the particle has to lose some kinetic energy}
\]

\[
\text{A reaction force from the radiation field acts back on the particle}
\]

Let's designate the radiation force as \( \vec{F}_{\text{rad}} \) and the power associated with this radiation force is

\[
\vec{F}_{\text{rad}} \cdot \vec{v}
\]

Assume that we can find two time marks \( t_1 \) and \( t_2 \) at which the state of the system is identical. Namely, same velocity fields and acceleration, then the total energy lost from the system is in the form of radiation,
\[ U_{\text{rad}} = \int_{t_i}^{t_f} P_{\text{rad}} \, dt = \int_{t_i}^{t_f} \frac{m_0 g^2 a^2}{6\pi c} \, dt \]

Meanwhile, the total work done by the radiation force is
\[ U_{\text{reaction}} = \int_{t_i}^{t_f} F_{\text{rad}} \cdot \vec{v} \, dt \]

From energy conservation, \[ U_{\text{reaction}} = -U_{\text{rad}} \]
\[ \therefore \int_{t_i}^{t_f} F_{\text{rad}} \cdot \vec{v} \, dt = -\frac{m_0 g^2}{6\pi c} \int_{t_i}^{t_f} a \, dt \]

where
\[ \int_{t_i}^{t_f} a \, dt = \int_{t_i}^{t_f} \frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt} \, dt \]
\[ = \int_{t_i}^{t_f} \frac{d\vec{v}^2}{dt^2} \cdot d\vec{v} \]

Integration by parts
\[ \int_{x_i}^{x_f} f(x) \, dx = \left[ f(x) \cdot \frac{d\vec{v}}{dt} \right]_{x_i}^{x_f} - \int_{x_i}^{x_f} \frac{d\vec{v}^2}{dt^2} \cdot d\vec{v} \]

since we assumed \[ \vec{v}_{t_i} = \vec{v}_{t_f} \] and \[ \vec{a}_{t_i} = \vec{a}_{t_f} \]
\[ \therefore \int_{t_i}^{t_f} a \, dt = 0 - \int_{t_i}^{t_f} \left( \frac{d\vec{v}^2}{dt^2} \cdot \vec{v} \right) \, dt \]

Hence
\[ \int_{t_i}^{t_f} F_{\text{rad}} \cdot \vec{v} \, dt = \frac{m_0 g^2}{6\pi c} \int_{t_i}^{t_f} \left( \frac{\vec{a}}{\vec{v}} \right) \cdot \vec{v} \, dt \]
\[ \Rightarrow \int_{t_i}^{t_f} \left( F_{\text{rad}} - \frac{m_0 g^2}{6\pi c} \frac{\vec{a}}{\vec{v}} \right) \cdot \vec{v} \, dt = 0 \]

The simplest answer (A sufficient condition, but not a necessary condition) is
\[ F_{\text{rad}} = \frac{m_0 g^2}{6\pi c} \frac{\vec{a}}{\vec{v}} \]

Abraham–Lorentz formula
These concepts become more clear when we solve the homework problems.

Hint

[11.13] Remember total radiated energy = \( \int_0^T P_{\text{rad}} \, dt \)

where \( T \) is the time span during which the particle decelerates from the initial speed \( v_0 \) down to zero.

Calculate the ratio between this term and the initial kinetic energy, which is \( \pm m v_0^2 \).

[11.17] The additional force \( F_e \) is used to counteract the radiation reaction, hence

\[
\frac{\text{d}F_e}{\text{d}t} = -F_{\text{rad}}.
\]

Need to calculate \( \frac{\text{d}a}{\text{d}t} \) for circular motion.

Remember

\[
\ddot{a} = \frac{v^2}{R} (-\ddot{R}) = -\frac{v^2}{R^2} \ddot{R}
\]

so

\[
\ddot{a} = -\frac{v^2}{R^2} \ddot{R}, \text{ when } v \text{ and } R \text{ are constants}
\]

\[
\frac{v^2}{R} = \frac{\ddot{R}}{\dot{R}} = \frac{R^2 - \dot{R}^2}{\dot{R}^3} = \frac{1}{v}
\]

The power \( P_e \) this extra force delivers is

\[
P_e = F_e \cdot \dot{v}
\]

For part (b), make sure to include explicit time-dependent terms in the final answers, namely \( \cos(\omega t) \), \( \sin^2(\omega t) \) terms.

And again compare \( P_e \) against \( P_{\text{rad}} \).
[11.19] The equation \( \dot{\mathbf{A}} = \tau \mathbf{a} + \frac{\mathbf{F}}{m} \) comes from
\[
ma = F_{\text{real}} + F
\]
\[
= \frac{\mu_0 q^2}{8\pi c} \mathbf{a} + F
\]
\[
a = \frac{\mu_0 q^2}{8\pi mc} \mathbf{a} + \frac{\mathbf{F}}{m} = \tau \mathbf{a} + \frac{\mathbf{F}}{m},
\]
where \( \tau = \frac{\mu_0 q^2}{8\pi mc} \).

(a)
\[
\int_{t_0}^{t+\epsilon} a \, dt = \int_{t_0}^{t+\epsilon} \tau \dot{a} \, dt + \int_{t_0}^{t+\epsilon} \frac{F}{m} \, dt
\]
or
\[
\int_{t_0}^{t+\epsilon} \frac{\mathbf{V}}{\dot{t}} \, dt = \tau \int_{t_0}^{t+\epsilon} \frac{\mathbf{a}}{\dot{t}} \, dt + \frac{\mathbf{F}}{m} \int_{t_0}^{t+\epsilon} \mathbf{F} \, dt
\]

(b) Solve differential equations such as the one above equation (11.81), under different conditions for (i) \( t < 0 \), (ii) \( 0 < t < T \) and (iii) \( t > T \). Leave all necessary constants in the answers (You should have 3 unknown constants)

(c) \( \mathbf{A} \) is continuous (conclusion from part (a)).
so at \( t = 0 \), \( a(t < 0) = A(t > 0) \)
and at \( t = T \)
\[
A(t < T) = A(t > T)
\]
Therefore you have 3 constants from part (b), but you also have two equations. Solve for the two other constants in terms of the third one.

Suppose you have constant \( A \) for (i) \( t < 0 \)
\( B \) for (ii) \( 0 < t < T \)
and \( C \) for (iii) \( t > T \)
Solve \( A \) & \( C \) and express them in terms of \( B \).
(d) Remember \[ V(t) = \int \alpha(t) \, dt \]

Again use boundary conditions such as \( V(-\infty) = 0 \), and \( V \) has to be continuous at both \( t=0 \) and \( t=T \) to determine all unknown constants raised by integration.

(e) Sketch based on formulas from parts (c) & (d).

No need for computer graphics. You can guess the exponential-looking curves for the charged particle.

Make sure to include results (straightforward) of an uncharged particle as well.