Practice Session — Getting to Know Dipole Radiations

First Hint for HW #6 (Due Monday Oct. 22, 2001)

[11.3] Radiation resistance $R$ is defined through the relationship of 

$$<P_{\text{loss}}> = <I^2 R> = <P_{\text{rad}}>,$$

namely, the power radiated out can be thought of as that lost to heat.

Use eq (11.22) for $<P_{\text{rad}}>$.

Don’t forget to take time average of $I^2 R$, since $I$ can be dependent on $t$.

[11.9] At $t=0$, the dipole moment of the ring is 

$$\vec{p}_0 = \int \lambda \vec{r}' \, d\ell'$$

As the ring rotates, 

$$\vec{p}(t) = p_0 \left[ \cos(\omega t) \, \hat{y} - \sin(\omega t) \, \hat{x} \right]$$

Find $\vec{p}$.

[11.12] Ring is neutral, so $V = 0$.

Need to find $\vec{A}(\vec{r}, t)$.

Follow the class notes of Oct. 15, Page 1

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \int \left[ \vec{J}(\vec{r}', t_0) + \frac{\partial}{\partial t} \left( \vec{r}, t_0 \right) \left( \frac{\hat{r} \cdot \hat{r}'}{c^2} \right) \right] d\ell'$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r} \left[ \int \vec{J}(t_0) d\ell' + \rho \vec{I}(t_0) \left( \frac{\hat{r} \cdot \hat{r}'}{c^2} \right) d\ell' \right]$$
\[ \mathbf{A}(\mathbf{r}, t) = \frac{M_0 \mathbf{I}(t_0)}{4\pi cr} \oint (\mathbf{r} \cdot \mathbf{r}') \, d\mathbf{r}' \]

Express \( \mathbf{r}', \mathbf{r} \), and \( d\mathbf{r}' \) all in terms of \( \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}} \)

and carry out the integral.

You can expect that the final answer for \( \mathbf{A}(\mathbf{r}, t) \) will have \( \mathbf{g} \) in it, since that's the direction of current is flowing.

Find \( \mathbf{E} \) \& \( \mathbf{H} \) from \( \mathbf{A} \), in spherical coordinates.

Finally, remember \( |\mathbf{m}| = I \cdot \ell \).

\[
\therefore m(t_0) = \pi b^2 I(t_0)
\]

[11.21] (c) Dipole moment

\[ \mathbf{P}(t) = Q \mathbf{d} \cos(\omega t) = P_0 \mathbf{d} \cos(\omega t) \]

From Eq. (11.21), find \( \langle \mathbf{s} \rangle \).

The power per unit area of floor is

\[ \langle \mathbf{s} \rangle \cdot \hat{z} \]

and remember

\[ \sin \Theta = \frac{R}{R} = \frac{R}{\sqrt{R^2 + h^2}} , \quad \cos \Theta = \frac{h}{\sqrt{R^2 + h^2}} \]

(b) \[ P_{\text{floor}} = \int \langle \mathbf{s} \rangle \cdot \hat{z} \, dA \]

\[ = \int_0^\infty \langle \mathbf{s} \rangle \cdot \hat{z} \cdot 2\pi R \, dR \]

(c) Potential energy \( U(t) = \frac{1}{2} K x(t)^2 \)

\[ \frac{dU}{dt} = -\left. \langle P_{\text{total}} \rangle \right|_{\text{ed}} \quad \text{energy loss in U per unit time} = \text{total radiated power} \]
Additional problems.

Example. An electron is released from rest and falls under the influence of gravity. In the first centimeter, what fraction of the potential energy lost is radiated away?

Assume the electron position is \( y(t) \).

So \( \vec{v} = - e \frac{y(t)}{y} \)

\( y(t) = \frac{1}{2} gt^2 \)

\( \Rightarrow \vec{p} = -\frac{1}{2} get^2 \hat{y} \)

And \( \vec{F} = -ge \hat{y} \)

From Eq. 11.6 a: \( P_{nd} = \frac{m_0}{6\pi c} \vec{F} \cdot \vec{p} \)

\( = \frac{m_0}{6\pi c} (ge)^2 \) \( \frac{t^2}{2} \) \( \sqrt{\frac{2h}{g}} \)

It is a constant of time.

For a full distance \( h \), the time it takes is \( t = \sqrt{\frac{2h}{g}} \)

Total energy radiated during the full fall is \( P_{nd} \cdot t = \frac{m_0}{6\pi c} (ge)^2 \cdot \sqrt{\frac{2h}{g}} \)

And the potential energy lost is \( mgh \).

Hence the fraction is \( \frac{m_0 (ge)^2 \sqrt{\frac{2h}{g}}}{mgh} \)

\( = \frac{m_0 \sqrt{\frac{2h}{g}}}{6\pi c} \)

\( = \frac{(4\pi \times 10^{-7}) \times (1.6 \times 10^{-19})^2}{6 \pi (9.1 \times 10^{-31}) (3 \times 10^8) \sqrt{0.01}} \approx 4 \times 10^{-22} \)

Therefore, almost all potential energy is converted to the kinetic energy, very little goes into radiation.
Example II. When a charged particle approaches a conducting surface, radiation is emitted, associated with the changing electric dipole moment of the charge and its image.

\[ \mathbf{p}(t) = 2q \mathbf{E}(t) \]

\[ \mathbf{\ddot{p}} = 2q \mathbf{\ddot{E}} \]

However, \( \mathbf{\ddot{E}} = \mathbf{F} = -\frac{1}{4\pi\varepsilon_0} \frac{\mathbf{q}^2}{(2\varepsilon)^2} \) (charge accelerates due to the attractive force from its image)\)

\[ \therefore \mathbf{\ddot{E}} = -\frac{1}{4\pi\varepsilon_0} \frac{\mathbf{q}^2}{4m^2} = -\frac{\mu_0 c^2 \mathbf{q}^2}{16\pi m^2} \quad (c^2 = \frac{1}{\varepsilon_0 \mu_0}) \]

Hence \( \mathbf{\ddot{p}} = -\frac{\mu_0 c^2 \mathbf{q}^2}{2\pi \varepsilon_0 m^2} \)

Power radiated:
\[ \mathbf{P}_{\text{rad}} = \frac{\mu_0 \mathbf{p}^2}{6\pi\varepsilon_0} = \frac{\mu_0}{6\pi\varepsilon_0} \left( \frac{\mu_0 c^2 \mathbf{q}^2}{8\pi m^2} \right)^2 \]

\[ = \frac{\mu_0^3 c^4 q^6}{6(4\pi)^3 m^2} \]

Example III. The magnetic north pole of the Earth does not coincide with the geographic north pole; in fact, it's off by ~1°. Relative to the fixed axis of rotation, therefore, the magnetic dipole moment of the Earth changes with time. So what is the Earth's magnetic dipole radiation?
Assume Earth's magnetic moment is $M$, and $\omega$ the angular velocity of Earth rotation,

\[
\vec{m}(t) = M \cos \psi \hat{x} + M \sin \psi \left[ \cos(\omega t) \hat{x} + \sin(\omega t) \hat{y} \right]
\]

\[
\vec{\ddot{m}}(t) = M \sin \psi (-\omega^2) \left[ \cos(\omega t) \hat{x} + \sin(\omega t) \hat{y} \right]
\]

\[
|\vec{\ddot{m}}(t)| = M \omega^2 \sin \psi
\]

From the class notes of Oct. 17 Page 3

\[
P_{\text{rad}} = \frac{M_0}{8\pi c^3} \vec{m}^2
\]

\[
= \frac{M_0}{8\pi c^3} M \omega^4 \sin^2 \psi
\]

\[
= \frac{\mu_0 M \omega^4 \sin^2 \psi}{6\pi c^3}
\]

The magnetic field of a magnetic dipole is \((\ref{eq:bfield}) \text{ of Griffiths}\)

\[
\vec{B} = \frac{\mu_0 M}{4\pi r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right)
\]

The Earth's magnetic field is $\approx \frac{1}{3}$ Gauss at the equator,

$\theta = 90^\circ$

\[
\vec{B} = \frac{\mu_0 M}{4\pi R^3}
\]

where $R$ is the Earth radius.

\[
M = \frac{4\pi R^3}{\mu_0} \vec{B} = \frac{4\pi (6.4 \times 10^6)^3}{4\pi \times 10^{-7}} (5 \times 10^{-5})
\]

\[
= 1.3 \times 10^{23} \text{ (A m}^2\text{)}
\]
Hence the total power radiated from the Earth’s magnetic dipole is:

\[
P_{\text{rad}} = \frac{(4\pi \times 10^7)(1.3 \times 10^{23})^2 \sin^2(110)}{6 \pi (3 \times 10^8)^3 \left(\frac{2\pi}{24 \times 60 \times 60}\right)^4}
\]

\[= 4 \times 10^{-5} \text{ (Watt)}\]

A finite value, but not alarmingly large.