
Outline
- 4-vectors and 4-dimension calculus
- Electromagnetic field tensor

Current density 4-vector \( J^\mu = (c\rho, \vec{J}) \)
Potential 4-vector \( A^\mu = (V, \vec{A}) \)

and we derived
\[
\Box A^\mu = -\mu_0 J^\mu \quad (\mu_0 \text{ is a fundamental constant})
\]

where
\[
\Box = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

Obviously,
\[
\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial (ct)} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}
\]

\[
= -\frac{i}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}
\]

and
\[
\frac{\partial}{\partial x^\nu} = \frac{i}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}
\]

For example,
\[
\frac{\partial J^\mu}{\partial x^\nu} = \sum_{\mu=0}^{3} \frac{\partial J^\mu}{\partial x^\mu}
\]

simplified notation for summation convention

\[
\frac{\partial J^\mu}{\partial x^\mu} = \frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3}
\]

\[
= \frac{\partial (ct)}{\partial (ct)} + \frac{\partial \vec{J}}{\partial \vec{x}} + \frac{\partial \vec{J}}{\partial \vec{y}} + \frac{\partial \vec{J}}{\partial \vec{z}}
\]

\[
= \frac{\partial \vec{J}}{\partial \vec{t}} + \nabla \vec{J} = 0 \quad \text{(from continuity equation)}
\]

Therefore \( \frac{\partial J^\mu}{\partial x^\mu} \) is now transformed to an equivalent equation
\[
\frac{\partial J^\mu}{\partial x^\mu} = 0
\]

When we derived the equation \( \Box A^\mu = -\mu_0 J^\mu \), we used
the Lorentz gauge condition, namely
\[ \nabla \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \quad \text{(Lorentz gauge)} \]

Rewriting this equation:

\[ \frac{1}{c^2} \frac{\partial V}{\partial t} + \nabla \cdot \vec{A} = 0 \]

or

\[ \frac{\partial V/c}{\partial t(x)} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 \]

Apparently, this is

\[ \frac{\partial A^\mu}{\partial x^\mu} = 0 \]

(Note, Griffiths' equation (12.135) is not correct.)

To summarize the fundamental law of the electromagnetic field:

\[ \square^2 A^\mu = -\mu_0 J^\mu, \quad \text{with} \quad \frac{\partial A^\mu}{\partial x^\mu} = 0 \]

\[ \partial J^\mu / \partial x^\mu = 0 \]

Why do these fundamental equations look so simple?

After all, the Maxwell's equations and all experimental observations are invariant under the Lorentz transformation, for which the 4-vector calculus & geometry were invented.

Of course, it turns out that all of the laws of physics are invariant under the Lorentz transformation. And these simple notations are applicable to every law of physics.

Finally, \( \vec{E} \) & \( \vec{B} \) fields.

Remember \( \begin{cases} \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \nabla \times \vec{A} \end{cases} \)

So

\[ B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = F_{23} \]

(\( F \) is apparently a 4x4 matrix, with rows from 0 to 3 and columns from 0 to 3)
and \[ b_y = \frac{\partial A_x}{\partial x} - \frac{\partial A_z}{\partial y} = F_{31} \]
\[ b_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = F_{12} \]

By definition, \[
\begin{align*}
F_{13} &= -F_{31} \\
F_{12} &= -F_{21} \\
F_{11} &= -F_{12}
\end{align*}
\]

\[
E_x = -\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} \\
\Rightarrow \frac{E_x}{c} = -\frac{\partial A_x}{\partial (ct)} - \frac{\partial (V_x)}{\partial x} \\
&= +\frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = F^{01} \quad (F^{10} = -F^{01})
\]

\[
E_y/c = -\frac{\partial (V_y)}{\partial y} - \frac{\partial A_y}{\partial (ct)} \\
= \frac{\partial A^2}{\partial x_0} - \frac{\partial A^0}{\partial x_2} = F^{02} \quad (F^{20} = -F^{02})
\]

\[
E_z/c = -\frac{\partial (V_z)}{\partial z} - \frac{\partial A_z}{\partial (ct)} \\
= \frac{\partial A^3}{\partial x_0} - \frac{\partial A^0}{\partial x_3} = F^{03} \quad (F^{30} = -F^{03})
\]

The matrix \( F \) is antisymmetric, so the diagonal elements = 0
\[ F_{uu} = -F_{uu} \]
if \( u = v \), \[ F_{uu} = -F_{uu} \Rightarrow F_{uu} = 0 \]

Put everything together:
\[
F_{uv} = \begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
-E_x/c & 0 & B_z & -B_y \\
-E_y/c & -B_z & 0 & B_x \\
-E_z/c & -B_y & -B_x & 0
\end{pmatrix}
\]

This is the electromagnetic field tensor.
HW #11 Hints

[12.45] Use equation (12.102) for calculation of $\vec{E}$.

Here we have $\theta = 90^\circ$, $\vec{R}' = \vec{y}$, $R' = d$

In each system $A$, $B$, $C$, find how fast the (-8) charge is moving, and determine the respective $v_A$, $v_B$, $v_C$, along with $V_A$, $V_B$, $V_C$.

For the $\vec{B}$ field, use the expression $\vec{B} = \frac{1}{c^2} \vec{V} \times \vec{E}$

($\vec{V}$ is the particle velocity)

Finally, the force in each frame is given by

$\vec{F}' = \vec{F} \left( \vec{E} + \vec{V}_{(B)} \times \vec{B} \right)$

You can check Eq. (12.68) to see how the force is transformed among the three different frames.

[12.46] Use equation (12.108) of field transformation, determine that

$\vec{E} \cdot \vec{B} = \vec{E}' \cdot \vec{B}'$

and $E^2 - c^2 B^2 = E'^2 - c^2 B'^2$

For part (c), remember $E^2 - c^2 B^2$ is invariant. So if it is positive in one frame, it will remain positive in any other frame.

[12.47] (a) Go back to chapter 9, especially equations (9.51) & (9.52).

Understand the concept of polarization.

(You can assume $\delta = 0$, initial phase $= 0$)

Understand the relationship between $\omega$, $k$, $c$, $\lambda$, etc.
(b) Use field transformation (12.108) to find amplitudes.

\[ \begin{align*}
\xi &= \eta (\xi + \nu \xi) \\
\eta &= \eta (\xi + \frac{\nu}{c^2} \xi)
\end{align*} \]

to express \((kx - \omega t)\) in terms of \((k\xi - \omega \eta)\).

Determine \(\xi\) & \(\eta\)

\(\begin{align*}
\xi &= \text{proportional factor} \cdot k \\
\eta &= \text{proportional factor} \cdot \omega
\end{align*} \)

This is the relativistic Doppler effect.

(c) Remember intensity \(\propto E^2\)

[12.63] (a) From example 5.8 on page 227

\[ B' = -\frac{u_0}{c} \cdot \vec{K} \cdot \vec{y} \]

Then Torque \(\vec{N} = \vec{m} \times \vec{B}\)

(b) If \(\xi = \sigma \vec{v}\), and \(|\vec{m}| = \lambda V \lambda^2\)

plug these quantities into \(\vec{N}\) and find its magnitude.

In the moving frame \(\bar{S}\)

\[ \begin{align*}
\vec{v} &\quad \vec{u} \quad \vec{v} \\
(12) &\quad (a) &\quad (b)
\end{align*} \]

charges on the front side (a) are at rest.
and the density \((\lambda_0)_a = \frac{\lambda}{\gamma} \quad \gamma = \frac{1}{\sqrt{1-\nu^2}}\)

(\(\lambda\) is the charge density in the original \(S\) frame)

In \(\bar{S}\), the length of (a) is contracted, to \(\frac{L}{\gamma}\)

so charge on side (a) is \((\lambda_0)_a^\prime = \frac{\lambda}{\gamma^2}\)

For the back side, \((\lambda)_b^\prime = \frac{2\nu}{1+\nu^2} \cdot \text{find } (\lambda)_b\)

\[ (\bar{\xi})_b = (\bar{\eta})_b (\lambda_0)_a \]
and so the net charge on side (b) is \( (x_0 \cdot \frac{l}{2}) \).

(The charge density and length for sides (c) and (d) don't matter.)

From this charge imbalance between side (a) & (b), determine the dipole moment \( \vec{p} \).

The surface charge density \( \sigma_0 = \frac{\sigma}{l} \) in \( S \) (rest frame).

\( \sigma \) is the surface charge density in \( S \).

\[ \vec{E} = \frac{\sigma_0}{2\varepsilon_0} \vec{E} \]

Find the torque on the electric dipole \( \vec{J} = \vec{p} \times \vec{E} \).

[(12.64)] You can set up the following configuration, for example

\[ \hat{x} \quad \hat{y} \quad \hat{z} \]

\[ \vec{E} \quad \vec{B} \]

So \( \vec{E} \) & \( \vec{B} \) are neither parallel, nor perpendicular.

And \( \vec{E} = (0, 0, E) \), \( \vec{B} = (0, B \cos\phi, B \sin\phi) \)

Go to a frame moving at speed \( v \) along the \( \hat{x} \) direction.

Find \( E_x, E_y, E_z \) and \( B_x, B_y, B_z \) in the new frame.

To have \( \vec{E} \) and \( \vec{B} \) parallel, you need

\[ \frac{E_y}{E_z} = \frac{B_y}{B_z} \]

From this relation, you can reach the final expression given by the problem.