I. The nature of superpositions

A. Consider the following student’s statement about a quantum mechanical infinite square well where $\psi_1$ and $\psi_2$ are the ground state ($n = 1$) and first excited state ($n = 2$), and each satisfies the time-independent Schrödinger equation. (i.e., $H\psi_1 = E_1\psi_1$, and $H\psi_2 = E_2\psi_2$.)

“The wave function given by $\Psi(x,0) = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2$ represents a lack of knowledge about the state of the system. The system is definitely in either the ground state or the first excited state. The wave function simply tells you that the probability is 1/2 that the system is really in the ground state and 1/2 that it is really in the first excited state.”

Do you agree or disagree with this statement?

B. Suppose you have the following two ensembles of systems in a quantum mechanical infinite square well at time $t = 0$:

1) half of the systems are in state $\psi_1$ and half of the systems are in state $\psi_2$

2) all of the systems are in state $(\psi_1 + \psi_2)/\sqrt{2}$.

At $t = 0$ would the probability of finding a particle on the left half of the well in a randomly selected system from ensemble (1) be greater than, less than, or equal to the probability of finding a particle on the left half of the well in a randomly selected system from ensemble (2)? Would your answer change as time evolves? Explain.

C. Check that your answers to parts A and B are consistent. Explain.
II. Eigenstates of the square well in the harmonic oscillator potential.

Consider a quantum mechanical harmonic oscillator. This system is prepared so that at $t = 0$ it is in a state described by a wave function identical to that of the ground state of the infinite square well. That is, $\Psi(x,0) = \sqrt{1/L} \cos(\pi x / 2L)$ between $-L$ and $L$.

A. Does the probability density associated with this state depend on time? If so, describe its time dependence. If not, explain why not.

B. Consider the following student’s statement.

“This state does depend on time. The full wave function is given by $\Psi(x,t) = \sqrt{1/L} \cos(\pi x / 2L)e^{-iEt/\hbar}$. The probability density, however, does not depend on time since the phase cancels out upon squaring.”

Do you agree or disagree? Explain.

Does this student’s wave function satisfy the Schrödinger equation for the harmonic oscillator? Show and explain your work.

C. Is the initial state of the harmonic oscillator shown above, $\Psi(x, 0)$, associated with a single energy? Explain. (Hint: Consider the time-independent Schrödinger equation?)

Is $\Psi(x, 0)$ a valid state in the harmonic oscillator? If so, describe how you would write the full time dependent wave function $\Psi(x, t)$.

✔ Discuss your answers with an instructor before continuing.
III. Energy Eigenstates

Again consider a quantum mechanical harmonic oscillator.

A. Assume that energy measurements on this oscillator in a particular state could only yield $E_o$ or $E_1$ with equal probability. Write and sketch $\Psi(x, 0)$. If it is not possible to completely specify $\Psi(x, 0)$ with the given information, write and sketch at least two possible $\Psi(x, 0)$ consistent with the given information.

B. Now assume the quantum mechanical harmonic oscillator is prepared so that its initial state is given by:

$$\Psi_i = \Psi(x, 0) = i \sqrt{\frac{1}{3}} \psi_0 - \sqrt{\frac{2}{3}} \psi_1$$

where $\psi_0$ and $\psi_1$ are the ground state ($n = 0$) and first excited state ($n = 1$), and each satisfies the time-independent Schrödinger equation. (i.e., $H \psi_o = E_o \psi_o$, and $H \psi_1 E_1 \psi_1$.)

1. Sketch a graph of each term in the linear combination that constitutes $\Psi(x, 0)$. Describe how a graph of $\Psi(x, t)$ would change with time.

2. Is the $\Psi_i$ an energy eigenstate? Justify your answer.

3. Suppose that at time $t = 0$ you measured the energy of the system.

   What value or values could a measurement of the energy yield?

   What is the probability that your energy measurement would yield the value $E_o$?
Is the probability above dependent on the time at which the measurement is made? Explain.

Explain how your answers about energy measurements are consistent with the time dependence of $\Psi(x, t)$ that you discussed above.

4. On the basis of your results above, can “the energy of the system” in this state be represented by a single number or is this quantity not well-defined?

C. Suppose that an energy measurement made on the system in its initial state $\Psi_i$ described above yields the first excited state energy, $E_1$.

1. Write down an expression for the state of the system immediately after the energy measurement is made.

2. Do you agree that acting on the initial wave function with the Hamiltonian corresponds to measuring the energy? Explain.

3. Imagine that a few minutes later you measured the energy of the system again. (Assume that, in the time interval between the first and second measurements, the system remained isolated from the environment.) On the basis of your results above, what would you expect the results of your energy measurement to be? Explain.

✓ Discuss your answers with an instructor before continuing.
IV. Summary
Assume $\Psi(x, 0)$ is a general quantum mechanical wave function describing the initial state of an isolated system with a time-independent potential.

A. Would the possible values for energy measurements on this system change as time evolves? Explain.

Would the relative probabilities of measuring these values change as time evolves? Explain.

B. Would the possible values for position measurements on this system change as time evolves? Explain.

Would the relative probabilities of measuring these values change as time evolves? Explain.

C. Describe how you would find the state of the system, $\Psi(x, t)$, as a function of position and time given the functional form of $\Psi(x, 0)$, and the functional form of the potential.