Balls on tracks
A ball rolls back and forth in a track with very steep sides. Two levels of unequal length joined by a steep ramp form the base of the track. A large number of photographs of the system are taken at random times. Assume that the time spent on the steep portions is negligible. Also assume there is no friction or energy loss in the system, and that the ball rolls smoothly, without bouncing, forever. Level 1 has a length of $3/4L$, and level 2 has a length of $L$. Assume the ball was dropped from a height of $4h$.

A. Probability

1. Sketch the gravitational potential energy corresponding to this situation between $x = 0$ and $x = 1.75L$.

2. Is the speed of the ball on level 1 greater than, less than, or equal to its speed on level 2? Explain.

For the ball, is the amount of time that it spends on level 1 greater than, less than, or equal to the amount of time it spends on level 2? Explain.

3. Suppose a single photograph were taken at a random time. On the basis of your results above, is the probability of the photograph showing the ball on level 1 greater than, less than, or equal to that of the photograph showing the ball on level 2? Explain.

4. Determine the probability of finding the ball on each level. Explain.

✓ Check your results with a tutorial instructor.
B. *Probability density*

Imagine splitting level 2 into two unequal segments: segment “2A” from \( x = 0.75L \) to \( x = L \), and segment “2B” from \( x = L \) to \( x = 1.75L \).

1. Find the probability that out of anywhere in the system the ball is found:
   
a. along segment 2A
   
b. along segment 2B

   Explain your reasoning.

Consider the following ratio for each segment: The probability of finding the ball along that segment divided by the length of that segment.

2. Is the above ratio larger for segment 2A, larger for segment 2B, or is it the same for both segments? Explain.

The ratio defined above is called *probability density*.

3. Compare and contrast *probability density* with other densities that you have encountered in physics. What are the units of probability density in this case?
In the space at right, carefully draw a graph of probability density, $\rho(x)$, versus position from $x = 0$ to $x = 1.75L$. Label relevant values on the vertical axis.

What feature of the graph represents the probability of finding the ball in an arbitrarily chosen interval between $x = x_1$ and $x = x_2$?

What is the probability of finding the ball exactly at $x = L$? Explain.

What answer would you expect for the probability of finding the ball anywhere between $x = 0$ and $x = 1.75L$? Show that your graph of probability density gives you the answer you expect.

Suppose you were given an arbitrary probability density function $\rho(x)$ (i.e., one that does not have a shape as simple as the one above). Write a mathematical expression for the probability of finding the ball between $x = x_1$ and $x = x_2$. 
C. **Averages**

Suppose a large number of photographs of the experiment on the first page are taken at random times, and the position of the ball in every photograph is measured. Write an expression that shows how the probability density, \( \rho(x) \), can be used to calculate the average of these positions.

Write an expression that shows how the probability density, \( \rho(x) \), can be used to calculate the average speed.

D. **Probability distributions as a function of time**

Suppose you know the position and velocity of the ball at \( t = 0 \). Sketch a reasonable probability distribution for the location of the ball at \( t = 0.5 \) s, and describe the behavior of this probability distribution as a function of time (i.e., describe \( \rho(x,t) \)). Explain.

How, if at all, is \( \rho(x,t) \) related to the probability distribution that you drew in section B4 above?

✓ Check your results with a tutorial instructor.