1. The definition of Hermitian operators. (10 points)

In class (and in Griffiths), we started by defining hermitian operators as those that yield real-valued expectation values. In other words, Hermitian operators are those where, given some arbitrary vector, $|\psi\rangle$, in the Hilbert space:

$$\langle \psi | \hat{O} \psi \rangle = \langle \hat{O} \psi | \psi \rangle$$

(1)

a) Use what you’ve learned about Dirac notation to explain why this equality follows from the requirement that the expectation values are REAL. (5 points)

In Eq. (1), the operator is used to transform $|\psi\rangle$, and the inner product between the original and transformed vector is then evaluated. However, hermitian operators are alternatively defined by a condition that appears to be much broader, involving TWO vectors of the space:

$$\langle \psi | \hat{O} \phi \rangle = \langle \hat{O} \psi | \phi \rangle$$

(2)

In fact, these two definitions of hermitian operators, Eqs (1) and (2) are completely equivalent. At least qualitatively, you recall that any particular state vector might be written as the superposition of several basis vectors, so Eq. (1) would certainly involve cross terms between the different basis vectors. Therefore, Eq. (1) must contain many specific terms that look like Eq. (2).

Let’s see how it works:

Imagine that you have two basis vectors, $|\alpha\rangle$ and $|\beta\rangle$, and that you can construct a third vector as a linear combination of them:

$$|\eta\rangle = |\alpha\rangle + |\beta\rangle$$

(3)

Similarly, let a second vector be a different linear combination, say this:

$$|\gamma\rangle = |\alpha\rangle - i|\beta\rangle$$

(4)

b) Show that definition (1) for $|\eta\rangle$ and then again, separately for $|\gamma\rangle$, implies definition (2) for $|\alpha\rangle$ and $|\beta\rangle$. (5 points)
2. **Eigen-value equations for two of our favorite Hermitian operators.** (20 points)

One of the important reasons that we emphasize Hermitian operators is that the real-valued expectation values of all observables must be related to Hermitian operators (see above) AND because Hermitian operators have eigen value equations that provide complete basis state vectors that we can use in expanding general state vectors in the Hilbert space.

In class, we considered the general case of some operator, \( Q \), and its eigen-value problem:

\[
\hat{Q}|q\rangle = q|q\rangle
\]

(5)

Here, we are encouraged to find the state vectors (the eigen vectors) such that when the operator transforms the vector you get back the same vector scaled by some real-valued constant (the eigen value for that particular eigen vector).

In this problem, we will look at the eigen value problem for two Hermitian operators, position and momentum. Our job is to solve the eigen-value problems using a variety of different representations of these operators.

a) **Momentum operator in the real-space representation.**

In the real-space representation for 1-dimensional problems along the x-axis, you know that the momentum operator is written as:

\[
\hat{p} = \frac{\hbar}{i} \frac{d}{dx}
\]

In the real-space representation, you also recognize that general wave functions (say evaluated at \( t = 0 \)) are to be written as functions of position. Hermitian operators are supposed to provide basis sets for expanding general state vectors, so what are the basis functions that come from the momentum operator?

Let’s call the basis functions something that reminds us of momentum, \( \Pi_p(x) \) and label each basis function with its momentum eigen value, \( p \). Then our job is to find these basis functions by solving the eigen value problem:

\[
\hat{p}\Pi_p(x) = p\Pi_p(x)
\]

Or

\[
\frac{\hbar}{i} \frac{d\Pi_p}{dx} = p\Pi_p
\]
Solve this eigen-value equation for the eigen functions and state whether you think that they look like a set of functions that can be used to expand general wave functions. Have you seen these before? (5 points)

b) Momentum operator in the momentum-space representation.

Now that you know the real-space representation of the momentum operators, show what they look like in the momentum-representation. Remember that a general real-space wave function, \( \psi \), is written as a superposition of the plane waves, and that the momentum space representation of that function is just the coefficients, \( \phi \):

\[
\psi_{\text{general}}(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{ipx/\hbar} \phi(p) dp
\]

where

\[
\phi(p) = \int_{-\infty}^{+\infty} e^{-ipx/\hbar} \psi_{\text{general}}(x) dx
\]

Pick one of the real-space momentum basis functions, say the one that goes with a particular value of momentum, \( p_0 \). What does the momentum space representation of your real-space momentum basis function for \( p_0 \) look like? Really do the transform above and state the momentum space function form. Then, comment on the eigen value equation in momentum space. Recall, in momentum space, the momentum operator is just multiplication by the momentum value. Then, the eigen value equation would look like this:

\[
p\Phi_{p_0}(p) = p_0 \Phi_{p_0}(p)
\]

Explain why this equation actually makes sense given your momentum space representation of the momentum eigen functions. (5 points)

c) Position operator in the momentum-space representation.

OK, now do the case of the position operator in the momentum representation. Here, we expect that the position operator is given by:

\[
\hat{x} = i\hbar \frac{d}{dp}
\]

Set up the eigen value problem and solve for the eigen functions of the position operator in the momentum representation. (5 points)

d) Position operator in the real-space representation.

And, finally, show what these functions look like in the position rep. (5 points):
3. **Functions as vectors tutorial**. (20 points)

Get together with a group of your peers and complete the Functions as Vectors tutorial that we worked on last Friday. You can find copies on the website if you need a fresh one. **Do all the pages.**