This problem set concentrates on angular momentum addition and two-level systems, and some of the interesting effects that we have ignored in the hydrogen atom, things that you will see more of in the 2nd semester.

1. **Addition of angular momentum.** (10 pts.)

   The last section in Griffiths Chapter 4 is about how to add angular momentum contributions, possibly from different particles in a composite system, possibly from orbital and spin angular momentum terms for a single particle. It’s useful to know how to think about angular momentum addition, because many of the systems we encounter actually are composed of multiple particles, say the way the hydrogen atom is composed of a proton and an electron, or the way the proton is composed of quarks, and because there are various possible sources of angular momentum. Recall that classical mechanics teaches us that it is the TOTAL angular momentum that is conserved as a vector quantity. All of these points suggest that addition of angular momentum could be important.

   a) **Do Griffiths Problem 4.34 on the sum of two spin \( \frac{1}{2} \) systems.** (5 points)
   
   For fun, spend 5 minutes on Google looking up Ortho and Para hydrogen and their properties. In the hydrogen MOLECULE, we have two protons. They can have their spins anti-aligned (parahydrogen) or else ‘aligned’ in a spin 1 triplet state. It turns out that the nuclear spin alignment can be very long-lived so that you can make essentially pure ortho and para hydrogen. They have different melting and boiling points, different heat capacities, etc. Very cool.

   b) **Do Griffiths Problem 4.35 on the possible spins of baryons and mesons.** (5 points)
   
   The end result of this problem is that some of these systems behave like FERMIONS (half integer spin) and some like BOSONS (integer spin). List which is which. Notice that in nature, integer spin particles are the force-carrying particles, like the photons (spin 1). In nuclear physics, the same is true. Which of the particles, baryons or mesons do you think might be carrying forces?

2. **Spin \( \frac{1}{2} \) linear combination states.** Griffiths Problem 4.49. (10 points)

3. **Spin \( \frac{1}{2} \) proton in a magnetic field.** More Nuclear Magnetic Spin resonance. (10 points)

   Consider the case of a spin \( \frac{1}{2} \) object in a magnetic field that points in the z-direction, as in the discussion in Griffiths on page 179-180. One common example of this type of system would be the protons inside hydrogen atoms, that are part of the water making up the chemicals in your body, during magnetic resonance imaging (MRI).

   a) When you place a spin in a magnetic field, the spin develops a high energy state and a low energy state. Look up the gyromagnetic ratio of the proton. A typical MRI machine puts you inside a solenoidal magnet with a magnetic field of 1.5
Tesla. What is the energy difference between these two states for a proton in this 1.5 Tesla field?

b) In the MRI process, spins from the low energy ‘aligned’ state are tipped 90 degrees into a state with, say, a pure eigen state of $S_x$. In the discussion on page 180, that means that $\sin \alpha = 1$ in Eq. 4.164. What is the expected frequency of oscillation for the spin components for a proton in a 1.5 Tesla field? How is this frequency of oscillation related to the energy difference that you calculated in part a)?

4) **Things we ignored in the Hydrogen Atom, things that you’ll see again:** (20 points)

Here at the end of the first semester, most of the problems that we have considered are exactly solvable. That’s why we have considered them! It’s the exactly solved problems where we have pretty direct information about every aspect of the solution. Of course, most problems are not exactly solved. For example, real potential wells can’t be perfectly square. Real springs are not perfectly Hooke’s Law for arbitrary stretch. Similarly, the hydrogen atom problem that we solved is idealized and exactly solved only for the ideal case. Let’s spend a short problem looking at some of the non-idealities and figure out how important them might be. All of these get more attention in the second semester where you learn how to approximately solve problems that cannot be solved in closed form.

a) **Center of mass solution.** Our solution in class assumed that the proton is at the center of the coordinate system and does not move. Under those circumstances, the predicted energies of the hydrogen atom are given by Griffiths equation 4.70:

$$E_n = -\left[ \frac{m}{2\hbar^2} \left( \frac{e^2}{4\pi \varepsilon_0} \right)^2 \right] \frac{1}{n^2}$$

Calculate the energy of the ground state using textbook values for the various constants of nature. Be sure to use numbers that are good to at least 4 decimal places and state the ground state energy to similar accuracy. (2 points)

This same combination of constants appeared in Bohr’s original model for hydrogen. However, spectroscopists knew the transition energies in hydrogen rather well, and one of them, Ralph Fowler (later Sir Ralph Fowler and thesis advisor of Dirac and several other Nobel Prize winners) pointed out that the transition energies were not quite right. Bohr quickly recognized his ‘mistake’ of not allowing the proton to move. Doing so effectively means solving the electron problem in the center of mass frame. That means the hamlitonian is exactly the same, but you need to use the ‘effective mass’ of:

$$\mu \equiv \frac{m_e m_p}{m_e + m_p}$$
in place of the electron mass. Put in numbers for the effective mass (again to at least three decimal places) and recalculate the ground state energy. This effect is one of the largest corrections to the energy. (3 points)

b) **Relativistic corrections to the kinetic energy.** Almost all of the remaining improvements to the hydrogen calculation require that we include the effects of relativity. Dirac’s relativistic wave equation for the electron does all of these things automatically, but you can see the approximate nature of these improvements by starting with the Schrödinger equation and thinking about the terms in either kinetic energy or potential energy, and making up stories about how relativity should change things.

For example, let’s think about the ‘kinetic energy’ piece of the Schrödinger equation:

\[
\frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m} \nabla^2
\]

In classical special relativity, the kinetic energy is rather more complicated:

\[
KE = \sqrt{p^2c^2 + m^2c^4} - mc^2
\]

You can see the critical role played by the relativistic rest mass energy, mc², and that it sets a natural scale for determining whether a particle is relativistic or not. For example, the rest mass energy of a single electron is around 511,000 eV. By comparison, you know that the ground state energy of the electron in hydrogen is part KE and part PE. The total of -13.6 eV turns out to be (see page 190-191 in Griffiths and the virial theorem) +13.6 eV of KE and -27.2 eV of potential energy, so the kinetic energy is clearly small compared to the rest mass. Therefore, we expect that we could do an expansion of the relativistic equation to get an approximate correction to the classical kinetic energy.

Expand the square root in the relativistic KE and show that you get:

\[
KE \approx \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}
\]

Use what you know about the estimated kinetic energy of the ground state and the rest mass of the electron (see above) to estimate the size of this correction to the ground state energy. (5 points)

c) **Relativistic corrections to the potential energy.** Then there is the question of the potential energy that we use. Is the simple coulomb potential really correct? To the extent that the electron moves and the proton carries a charge, you can appreciate that the proton appears to be a moving charge in the electron frame, so we might very well expect that there ought to be a magnetic field from the
perspective of the electron, and that the strength of the field might very well be

different for different orbitals, because the electron ‘motion’ might depend upon

the orbital.

Read Griffiths second 6.3.2 on “Sping-Orbit Coupling” and you’ll see a quick
classical argument about the effective magnetic field, that it does depend upon the
orbital angular momentum of the electron, and that (due to the electron spin) there
is a new term in the Hamiltonian, the spin-orbit term, that looks like:

\[ H_{so} = \left( \frac{\varepsilon^2}{8\pi\varepsilon_0} \right) \frac{1}{m^2 c^2 r^3} \vec{s} \cdot \vec{L} \]

Consider Griffiths Figure 6.9 for the \( n=2 \) state. You recall that for \( n=2 \), the
angular momentum quantum number can be \( l=0 \) or \( l=1 \). Further, the spin quantum
number can be only \( \frac{1}{2} \). Use these facts along with an estimate of the radius of the
\( n=2 \) orbital to estimate the splitting between the \( l=0 \) or \( l=1 \) quantum states. Be
careful here to double check that your units really work out to give you energy in
either eV or Joules. What rough frequency range does this splitting represent in
the electromagnetic spectrum? (5 points)

d) Other contributions. There are lots of additional contributions, many of which are
very important in understanding the details of atomic spectroscopy and many of
which were similarly important in motivating deeper understanding of quantum
mechanics and atomic physics.

Make a table that in two or three sentences briefly describes each of the following
and give at least one reference source, where an interested reader could learn more
for each.

Hyperfine splitting, Lamb Shift, Zeeman effect, Stark Shift, Nuclear penetration
effects. (5 points)