This homework set concentrates on the spin ½ system and the closely related two-level system. All of quantum mechanics is right here, without a single differential equation. Have fun!

1. Spin 1/2 system in a magnetic field. (10 points)

In this problem, we will consider the motion of a stationary spin ½ object in an applied magnetic field. Griffiths does the special case on pages 179-181 of the spin ½ object in a magnetic field that is entirely in the z-direction. Here, we’ll work through it again to be sure that you’ve really looked at it.

First, let’s lay out the basic facts, which are also reviewed in Griffiths’s section 4.4.1. in more detail. First, a spin ½ object, say an electron has a immutable (does not change) spin quantum number of \( s = \frac{1}{2} \). As is typical for quantum angular momentum problems, we choose to speak about the operators for the square of the spin, \( \hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \), and the z-component of angular momentum, \( \hat{S}_z \), which are operators that commute. The eigenvalues for \( \hat{S}_z \) are \( \pm \hbar/2 \) and \( \mp \hbar/2 \), while the eigenvalue for \( \hat{S}^2 \) is \( \hbar^2(s+1) = 3\hbar^2/4 \). We will use a representation for the operators, eigen vectors, etc. that is based on the 2x2 matrices, column vectors, row vectors, and simple rules of matrix multiplication. Let’s start by using the basis of the eigen vectors of \( \hat{S}_z \), written as column vectors as in Griffiths’s equations, 4.140 and 4.141:

\[
\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

In this representation, the operators for \( \hat{S}^2 \) and \( \hat{S}_z \) are then eqs. 4.143 and 4.145:

\[
\hat{S}^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

As Griffiths shows in the discussion on page 174, knowing about the raising and lowering operators, \( \hat{S}_+ \) and \( \hat{S}_- \), and understanding how they can move you from one basis function to the other, is particularly useful as they allow you to quickly work out both the matrix representation for themselves, and also for \( \hat{S}_x \) and \( \hat{S}_y \). The x- and y-spin components are then found to be:

\[
\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]
OK, so let’s go down the check list of the things that we normally expect to have if we’re going to describe quantum mechanical behavior of a system:

1) **Do we have a Hilbert space?** Yup! It’s a simple 2-d Hilbert space.

2) **Do we have a representation** for talking about states of the system? Yup! We have the orthonormal basis states of the Hermitian operators \( \hat{S}_z \) and \( \hat{S}_z^2 \).

3) **Do we have various operators** to describe the physical quantities of interest? Well, we’ve got lots of operators for the angular momentum properties. Also, we know that all operators in this representation will be 2x2 matrices. Looks pretty good!

4) **Do we have the Hamiltonian** so we can solve for the energy eigen vectors, the associated energy eigen values, so we can predict the time dependence of wave functions? Doh!! Biff!! Nope! So though we’ve made progress, we’re not ready to start talking physics. We still need the Hamiltonian! But still, notice how far we’ve come without even knowing the Hamiltonian! Not bad so far.

For the case of a stationary spin \( \frac{1}{2} \) particle, there is no kinetic energy to worry about. Further, even if it’s, say, a charged particle sitting in an electrostatic potential, since there is not going to be any moving around, we can just offset the energy scale so the potential is zero. What’s left is interesting new physics: Particles that carry intrinsic angular momentum also carry a proportional magnetic moment, \( \hat{m} = \gamma \hat{S} \), where the quantity \( \gamma \) is referred to as the ‘gyromagnetic ratio’. Further, a magnetic moment in a magnetic field has an energy of \( -\hat{m} \cdot \vec{B} = -\gamma \vec{B} \cdot \hat{S} \). Therefore, we have a Hamiltonian operator (assume that the magnetic field is just some constant vector, \( \vec{B}_0 \)) determined by the field dotted into the spin operator.

a) **The Hamiltonian for spin \( \frac{1}{2} \) in a magnetic field.** Assume a magnetic field that points in an arbitrary direction, so that it has x- y- and z-components. Use the spin matrices and work out what the Hamiltonian matrix is in the original basis of the eigen vectors of \( \hat{S}_z \). Double check that your matrix is Hermitian. (3 points)

b) **The energy basis states:** Now assume that the magnetic field points only in the z-direction and find the eigen vectors and eigen values of the Hamiltonian operator (there should be two of each). Make sure that your energy eigen vectors are orthonormal (normalize them if necessary, they should already be orthogonal). At this point, you have two exciting things: First, you now have the eigen vectors of the Hamiltonian. Further, you know the time dependence of these states. They each have a complex exponential time dependence, with an energy appropriate for each state. (2 points)
c) **Time dependence of the spin components.** Assume that you start the system with a wave function that is the eigen state of $\hat{S}_x$ for ‘spin-up’. In other words,

$$|\psi, t = 0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now consider the expectation values of the spin components as time evolves. In other words, calculate the three expectation values:

$$\langle \psi(t)|\hat{S}_x|\psi(t)\rangle = ? \quad \langle \psi(t)|\hat{S}_z|\psi(t)\rangle = ? \quad \langle \psi(t)|\hat{S}_z|\psi(t)\rangle = ?$$

Some of your may get additional insight into how to see what’s happening by writing the components in terms of the spherical coordinate angles $\theta$ and $\phi$. It’s not necessary, but you might like it. In which direction would you want the spin to be pointing to remove the time dependence? Write down at least one wave function with this property. All of these results will come back in various forms in the next couple of weeks. (3 points)

d) **Schrödinger’s Mouse.** ANY quantum system that only has two states can be mapped onto the spin $\frac{1}{2}$ system in a magnetic field. For example, we did the (FAT!) tutorial on Schrödinger’s Mouse. Let’s show how the mouse and various mouse operators map onto the spin $\frac{1}{2}$ problem: We have two properties, Mass and Happiness. Let’s let Mass be mapped onto $\hat{S}_z$. OK, so the eigen values are not the same as for the spin $\frac{1}{2}$ system, but the matrix is very similar. Let’s create a 2-d matrix representation where the basis states are those for mass and various operators are 2x2 matrices. In the mass representation, we’d have basis states of the form:

$$\chi_M = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and a mass operator that is then:

$$\hat{M} = \begin{pmatrix} 10m & 0 \\ 0 & m \end{pmatrix}$$

Write the ‘feed a mouse a pizza’ operator and the associated Hermitian adjoint, ‘suck a pizza out of a mouse’ operator. What operators do these correspond to in the spin system? (2 points)

2. **The Stern-Gerlach apparatus, state preparation, and uncertainty principles.** (30 pts.)
Griffiths describes the Stern-Gerlach apparatus on pages 181-183. Read that section and think about how exciting it must have been to see direct evidence that the angular momentum of atoms really does come in quantized units, as evidenced by the splitting of atomic beams into discrete numbers of beams. This process can be used to prepare atomic systems into well-defined angular momentum states. Also, the observation of the beam deflections, along with the known applied magnetic field gradient and initial atomic beam velocity, allows you to measure the angular momentum magnitude. That turns out to be a pretty good way to measure Planck’s constant.

The Stern-Gerlach apparatus is very useful when thinking about or discussing how quantum mechanics actually works. This problem asks you to consider using such an apparatus to make measurements, and to use what you know about quantum mechanics and spin ½ systems to work out some examples:

a) **Prepare an atomic beam.** In the original experiment of Otto Stern and Walther Gerlach (W. Gerlach and O. Stern, "Das magnetische Moment des Silberatoms". *Zeitschrift für Physik* 9: 353–355, (1922)), a beam of silver atoms is produced by boiling melted silver metal in an oven in a vacuum system. The oven has a small hole in the side that emits the atoms down a sequence of holes in sequential metal baffles, to produce an atomic beam. The atoms leave the oven with a thermal velocity distribution. Silver melts at 1234 K (pretty easy to remember). A typical atom leaves the oven with a velocity that you can estimate by assuming that the kinetic energy of the atom is equal to \( \frac{1}{2} k_B T \), where \( T \) is the melting temperature and \( k_B \) is Boltzmann’s constant of \( 1.38 \times 10^{-23} \) J/K. To do this calculation, you’ll need the mass of a typical silver atom. Silver is element 47 on the periodic table. **What is the typical silver atom velocity leaving the oven?** (5 points)

b) **Expected angular momentum of a silver atom.** (5 points) Silver comes in two naturally occurring stable isotopes, \(^{107}\text{Ag}\) and \(^{109}\text{Ag}\), with nuclei of 47 protons and respectively 60 and 62 neutrons. Both protons and neutrons are spin \( \frac{1}{2} \) particles and both tend to populate the nuclear states in pairs of opposite spin. Given this tendency, what is the total spin that you might expect for the nucleus of either isotope? Similarly, the electrons in the orbitals of the silver atom tend to occupy the orbitals in pairs of opposite spin. The last electron in the ground state configuration occupies a state of zero orbital angular momentum and, because the other electrons have spin and angular momentum contributions that just cancel, the total electronic spin is just \( s=1/2 \). Therefore, using the discussion of addition of angular momentum from Griffiths pages 184-188, **what are the expected total angular momenta (nuclear plus electronic) for the silver atoms?**

c) **Which spin matters?** In the problem above, you should have found that the total nuclear spin and the total electronic spin contributions are similar in size. However, the Stern-Gerlach apparatus separates the spin states of atoms based on the ELECTRONIC spin contribution. Notice that the apparatus uses a magnetic field gradient to act on the magnetic moments of the spins to apply a force to the atom. **Use the discussion on Griffiths page 178 regarding the ‘gyromagnetic ratio’**
that relates angular momentum to magnetic moment to argue that the spin of WHICH particle (electron or proton) dominates the force on the atom from the magnetic field and by roughly what numerical factor? (5 points)

d) Prepare an $s_z$ spin UP state. OK, so set the direction of the magnetic field and its gradient in the z-direction, so that the beam of silver atoms is split into two beams, one with z-component of spin that is spin up and one with spin down. Assume that the oven prepares atoms with completely randomized electron and nuclear spin directions (still in the lowest spin states, but pointed in random directions). What fraction of the beam do you expect to see in the spin UP beam? (5 points).

e) Other components of spin. Now take the stream of atoms that are in the $s_z$ UP beam and imagine that you measure the x-component of spin by using a second Stern-Gerlach apparatus appropriately oriented. Given that the initial beam of atoms is KNOWN to have $s_z = +\hbar/2$, calculate the probability for measuring $s_x = +\hbar/2$ or $s_x = -\hbar/2$ for the atoms going through the second apparatus. In other words, what fraction of the beam goes into the different directions from the second, x-directed, Stern-Gerlach? Note that you are asked to CALCULATE the probability. There is a crank to turn here. Don’t just insist on an answer. Rather, calculate the probability. (5 points)

f) What do you know now? At this point, you have measured the z-component and x-component of spin for the atoms. Your friend says, “Great! Now you know that the z-component of the atom’s spin is $s_x = +\hbar/2$ and that this beam also has $s_x = +\hbar/2$, while that beam has $s_x = -\hbar/2$. Therefore, we know two components of the spin angular momentum.” Do you agree? Explain your position and explain an experiment you could do to prove your point. (5 points).

3. Neutrino oscillations (thanks to K. T. Mahanthappa for discussion and a version of the calculation). (10 points)

You’ve reached the point where you have seen many of the essential models that physicists use for discussing quantum mechanical systems. The infinite potential well and the finite potential well are used to talk about any situation where you have particles in a box and the details are not believed to be important. You’ll see lots of examples in thermodynamics and statistical mechanics. The harmonic oscillator is used for any mass-on-a-spring system. It’s amazing how many systems of interacting particles are approximately described by the Hooke’s Law spring. Most recently, we’ve been looking at the hydrogen atom (the essential wavefunctions for describing the entire periodic table) and the associated angular momentum and angular momentum algebra.
The angular momentum algebra allowed us to discover the possibility of intrinsic angular momentum of both the integer and half-integer type. It turns out that the spin $\frac{1}{2}$ case, where there are only two allowed values of the $z$-component of the spin. When placed in a magnetic field, these two spin values are correlated with two different energy levels. Therefore, this spin $\frac{1}{2}$ model is used endlessly to model any system where there are only two known possible physical energy levels (with possibly other associated quantities: Think energy and associated spin). In this problem, we’ll look at one particular case. In the extra credit problem set we’ll discuss a couple of more. Here, we’ll discuss the topic of neutrino oscillations.

Have a look at the Wikipedia entry for the neutrino to get a balanced description of its history. Here’s a very quick overview without proper references: Neutrinos were first suggested in 1930 by Wolfgang Pauli to explain missing momentum and angular momentum in beta-decay of nuclei: In beta-decay, an electron is observed to be emitted by a decaying nucleus. After the decay, you can measure the electron momentum and the nuclear momentum and you find that momentum is not conserved. Pauli invoked the existence of a new particle, not detected during the decay, that carried the missing momentum. The lack of evidence of the particle in detectors suggested that it must be neutral. Similarly, the total spin angular was not conserved. Pauli therefore suggested that the particle must carry spin $\frac{1}{2}$. Taken together, Pauli suggested that we look for a neutral spin $\frac{1}{2}$ particle leaving such nuclear decay events. Because the neutrino interacts with matter largely through the weak force, it took nearly 25 years (C.L Cowan Jr., F. Reines, F.B. Harrison, H.W. Kruse, A.D McGuire (July 20, 1956). "Detection of the Free Neutrino: a Confirmation". Science 124 3212, (1956)) to actually detect it.

Our present understanding is that there are three types of neutrinos (and three associated antineutrinos), each associated with one of the three leptons. Thus, we have electron neutrinos, muon neutrinos, and tau neutrinos. These different types of neutrinos are labeled by their flavor e.g., the muon neutrino is said to be in a ‘muon flavor state’. Each flavor of muon shows up associated with decay of the particles that mediate the weak force, namely, the $W$ bosons. For example, muons are produced in cosmic ray events in the Earth’s atmosphere. The muon has a finite lifetime and then decays after roughly 2 microseconds into a $W$ and a muon neutrino. Later, the $W$ decays into an electron and an electron antineutrino. The tau lepton and the associated tau neutrino are extremely rare. Therefore, in natural events it is far more likely to see neutrinos of only two flavors: electron and muon. In essentially all other properties, the electron and muon neutrinos appear to be identical. Therefore, we can map the flavor basis onto the basis of spin UP and spin DOWN, and we've created a mapping of the electron and muon neutrinos onto a two-state model. Thus, upon creation in a radioactive decay event, a neutrino can be created in one of two possible flavor basis states. We call them:

$$\text{flavor basis } \Rightarrow \begin{cases} |\nu_e\rangle & \text{electron flavor} \\ |\nu_\mu\rangle & \text{muon flavor} \end{cases}$$
There is a lot of implied information here: For example, flavor is a measurable quantity. Therefore, the flavor states are assumed to be eigenstates of a hermitian flavor operator and are thus an orthonormal basis. These basis vectors are orthogonal.

Similarly, other observable neutrino properties can be discussed and imagined. Though neutrinos are general found to be ultra-relativistic particles, it’s conceivable that they have mass. Therefore, there should be a set of orthogonal mass basis states associated with the hermitian mass operator:

\[
\text{mass basis } \Rightarrow \begin{align*}
|v_1\rangle & \quad \text{eigenvalue, } m_1 \\
|v_2\rangle & \quad \text{eigenvalue, } m_2
\end{align*}
\]

You might be tempted to think of these states as analogous to the spin ½ eigenstates of \( s_z \), the \( |\uparrow\rangle \) and \( |\downarrow\rangle \) states, and for many years, people imagined that mass and flavor were compatible operators i.e., that they shared a set of eigenvectors, and that you could just select a single basis set like the spin up and spin down set.

Notice also, that mass is clearly related to energy via \( E = mc^2 \), so the mass basis would also be the energy basis.

To be concrete, let’s imagine a representation for the mass basis that uses two-component column vectors for the mass eigenvectors, and 2x2 matrices for all the operators. Set up a basis set for the mass states, so that:

\[
|v_1\rangle \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |v_2\rangle \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

In this representation, the mass operator is:

\[
\hat{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}
\]

In the rest frame of the neutrino, the Hamiltonian (since mass and energy are related as above) is then:

\[
\hat{H} = c^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}
\]

a) The Hamiltonian adjusted to a new energy scale and time-dependent wave functions. To make things look as much as possible like a problem that we’ve already seen, choose an energy scale such that the zero of energy is set at the average of the two rest-mass energies and rewrite this Hamiltonian operator in terms of this new scale. Compare your Hamiltonian to Griffiths equation 4.160 for a spin ½ particle in a magnetic field along the z-axis. Then, reproduce the
discussion on pages 179 and 180 to write down the time dependence of an arbitrary wave function that is written as a linear combination of the energy basis states. You should derive a version of equation 4.163. At this point, you have succeeded in mapping the neutrino mass states onto the problem of a spin \( \frac{1}{2} \) particle in a magnetic field. Pretty nice! (5 points)

b) **The time evolution of flavor states:** Starting in the late 1960s, researchers began to directly study electron neutrinos produced in the fusion reactions in the Sun. Hans Bethe’s Nobel Prize of 1967 came for explaining the Sun’s power source, and the fusion reactions were believed to be very well understood by that time. Nevertheless, the flux of electron neutrinos was regularly found to be roughly \( \frac{1}{2} \) to \( \frac{1}{3} \) of the expected flux. The experiments were sensitive to electron neutrinos, and it became clear that perhaps the electron neutrinos were being converted to muon and tau neutrinos during the travel time from the Sun to the Earth. In the language above, this type of behavior is an indication that the flavor operator and the mass operator are incompatible, so that the flavor basis and mass basis are not the same.

In this picture, the flavor basis vectors can be written as linear combinations of the mass basis vectors. Assume that the initial electron flavor vector is written as the linear combination, Griffiths equation 4.163 at \( t = 0 \). Note that the muon flavor vector is orthogonal to the electron flavor vector. Calculate the muon flavor vector. Finally, imagine that a fusion process in the Sun creates an electron neutrino at \( t = 0 \). Calculate the probability density and probability that you will observe the original electron neutrino as a muon neutrino at later times. (5 points)

In other words, calculate:

\[
\langle \nu_\mu | \nu_e(t) \rangle = ? \quad \text{and} \quad \left| \langle \nu_\mu | \nu_e(t) \rangle \right|^2 = ?
\]

If things have gone well, you have just shown that the electron neutrino can change into a muon neutrino (and oscillates back and forth between electron neutrino and muon neutrino) as time passes.