Exam 1 Closed book, one 8.5x11" cheat sheet, and calculator.  
Test time: 90 minutes  
Total points: 100

Please do not open the exam until you are asked to.

- Your exam should have 10 pages, numbered Exam 1.1 thru Exam 1.10.
- This exam consists of long-answer problems.
- Write your answers in the space provided on each page for the written question. Use the back of the page if needed. SHOW YOUR WORK ON THE WRITTEN QUESTIONS. Full credit will be given only for the correct answer accompanied by your reasoning.

PLEASE!!

1. Print your name on each page in the space provided.
2. Print your student Identification Number on the page above.

I recommend:
Don't waste time erasing on the written problems. Just put a line through defective reasoning. Ask for more paper if needed.
At the end of the exam, check that you have completed all the questions.

By handing in this exam, you agree to the following statement: "On my honor, as a University of Colorado Student, I have neither given nor received unauthorized assistance on this work"

Signature

Good luck!
Problem 1. **Foundations: Wave functions and operators.** (20 pts total)

a) In quantum mechanics, we expect to describe particle behavior via complex-valued wave functions. **State mathematically the ‘normalization condition’** we expect any single particle wave function to satisfy. **Then state what it means physically** (5 pts)

\[
\text{Given some } \psi(x,t) \\
\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1
\]

The probability of finding the particle somewhere is unity for any time, t.

b) Given a wave function, \( \psi(x,t) \), write down the equations you need to evaluate to find the expectation value of position and squared position. **State how these two values let you calculate the squared variance of position.** (5 points):

\[
\langle x \rangle = \int_{-\infty}^{\infty} \psi^* x \psi \, dx
\]

\[
\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^* x^2 \psi \, dx
\]

\[
\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2
\]
Problem 1.  Foundations: Wave functions and operators (continued)

c) Given a wave function, \( \Psi(x,t) \), write down the equations you need to evaluate to find the expectation value of momentum and squared momentum. State how these two values let you calculate the squared variance of momentum. (5 points).

\[
\langle p \rangle = \int_{-\infty}^{\infty} \Psi^* \frac{k}{i} \frac{d\Psi}{dx} dx
\]

\[
\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi^* \left[ -\frac{\hbar^2}{m} \frac{d^2\Psi}{dx^2} \right] dx
\]

\[
\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2
\]

d) How do you expect the variance of the position and variance of the momentum to be related for a typical quantum state? Explain. (5 points).

We expect \( \sigma_x^2 \sigma_p^2 \approx \hbar^2 \)

or

\( \Delta x \Delta p \geq \hbar \)

This expectation is based on the Uncertainty Principle.
Problem 2.  **Foundations: The Schrödinger Equation (20 points total)**

a) Write down the time dependent Schrödinger Equation for a 1-dimensional quantum problem (depends upon time, $t$, and position, $x$). State clearly the meaning of any constants or functions you use in your equation. (5 points)

$$i \hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x,t) \Psi(x,t)$$

- $\hbar$ = Planck's const.
- $m$ = particle mass
- $V(x,t)$ = Classical potential
- $i = \sqrt{-1}$

b) Write down the time independent Schrödinger Equation for a 1-d problem (depends upon position, $x$). State under what conditions you expect solutions of this equation to be appropriate for a problem, and under what conditions you must use the full time dependent Schrödinger Equation. If you use new symbols that you did not have in part (a), define them. (5 points):

**TISE:**

$$\frac{-\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + V(x) \Psi(x) = E \Psi(x)$$

This TISE is appropriate when $V$ is indep. of time. Otherwise, you use the full TDSE

$E$ = particle energy
Problem 2. Foundations: The Schrödinger Equation (continued)

c) Assume that you have solved the time independent Schrödinger Equation for the
eigen functions and eigen energies, $\Psi_n(x)$ and $E_n$. Write down the full space and
time dependent wave function for the state, $n$. (5 points)

$$\Psi_n(x,t) = \Psi_n(x) e^{-iE_n t / \hbar}$$

d) Assume that you have solved the time independent Schrödinger Equation for the
eigen functions and eigen energies, $\Psi_n(x)$ and $E_n$. Write down the full space and
time dependent wave function for a quantum state that satisfies general initial
conditions. If you use any new constants, state clearly either their value or how
you determine them from the initial conditions. (5 points)

$$\Psi(x,t) = \sum_n c_n \Psi_n(x) e^{-iE_n t / \hbar}$$

Orthogonality of the
stationary states allows
us to determine the $c_n$
Problem 3. The infinite square well. (30 points total)

a) You are asked to solve for the wave functions and eigen energies for the infinite square potential shown below. The solutions are of the form $A\sin(kx)$, so the wave function is zero at the $x = 0$ boundary. State the condition at $x = d$. State the allowed values of $k$. Write down an equation for the eigen energies. Sketch the third state at $t = 0$ on the picture below. (15 points)

\[ A\sin kd = 0 \text{ or } kd = n\pi \quad n = 1, 2, 3 \ldots \]

\[ k = \frac{n\pi}{d} \]

\[ E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} \left( \frac{n\pi}{d} \right)^2 \]
Problem 3. The infinite square well. (continued)

b) Assume that you prepare a more general state that is a superposition of the first and the third stationary state so that \( \psi(x, 0) = A[u_1(x) - iu_3(x)] \). Assuming that the stationary states are normalized already, calculate the normalization constant, \( A \). (10 points)

\[
\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1 \quad \text{Choose } A \text{ to make this true.}
\]

\[
\int_{-\infty}^{\infty} \psi^* \psi \, dx = A^2 \int_{-\infty}^{\infty} (u_1^* + iu_3^*)(u_1 - iu_3) \, dx = A^2 \left[ \int_{-\infty}^{\infty} u_1^* u_1 \, dx + i \int_{-\infty}^{\infty} u_3^* u_3 \, dx - i \int_{-\infty}^{\infty} u_1^* u_3 \, dx - \int_{-\infty}^{\infty} u_3^* u_1 \, dx \right]
\]

\[
= A^2 \left[ 1 + 0 + 0 + 1 \right] = A^2 \cdot 2 = 1
\]

So

\[
A^2 \left[ 1 + 0 + 0 + 1 \right] = A^2 \cdot 2 = 1
\]

\[
A = \frac{\pm 1}{\sqrt{2}}
\]
Problem 3. The infinite square well. (continued)

c) Using your normalized state from (b), and the actual form of the wave functions for the square well, calculate the expectation value of the squared momentum. (5 points)

\[
\langle p^2 \rangle = -\frac{\hbar^2}{2a^2} \int_{-\infty}^{\infty} \left[ u_1^* u_1 + u_3^* u_3 \right] \frac{d^2}{dx^2} \left[ u_1 - i u_3 \right] dx
\]

\[
\frac{d^2}{dx^2} u_1 = \frac{d^2}{dx^2} \sqrt{\frac{2}{a}} \sin \left( \frac{\pi x}{a} \right) = -\frac{2}{a} \left( \frac{\pi}{a} \right)^2 \sin \left( \frac{\pi x}{a} \right) = -\frac{\pi^2}{a^2} u_1
\]

\[
\frac{d^2}{dx^2} u_3 = \frac{d^2}{dx^2} \sqrt{\frac{2}{a}} \sin \left( \frac{3\pi x}{a} \right) = -\frac{2}{a} \left( \frac{3\pi}{a} \right)^2 \sin \left( \frac{3\pi x}{a} \right) = -\frac{9\pi^2}{a^2} u_3
\]

\[
\langle p^2 \rangle = -\frac{\hbar^2}{2} \int_{-\infty}^{\infty} \left[ u_1^* u_1 + u_3^* u_3 \right] \left[ -\frac{\pi^2}{a^2} u_1 - i \frac{9\pi^2}{a^2} u_3 \right] dx
\]

Use orthonormality again to get:

\[
\langle p^2 \rangle = \frac{\hbar^2}{2} \left[ \left( \frac{\pi}{a} \right)^2 + \left( \frac{3\pi}{a} \right)^2 \right]
\]

\[
= \hbar^2 \left( \frac{\pi}{a} \right)^2 \frac{5}{2}
\]
Problem 4.  The harmonic oscillator (30 points)

a) Think about a simple diatomic molecule like N$_2$. The atoms in the molecule are bound together by the sharing of electrons and the complicated dance of electronic potential and kinetic energies. Never the less, it’s a very good approximation to treat such molecules as though they were masses connected by a spring. Let’s assume that you can grab the atoms in a nitrogen molecule, and begin pulling them apart. As you do, the energy of the molecule increases because of the potential energy you put into the spring. Assume that the energy increases by 1 eV when you stretch the spring by 0.1 nanometers. What is the spring constant for this spring?  (10 points)

\[ \Delta \text{PE} = 1 \text{eV} = 1.6 \times 10^{-19} \text{ Joules} \]

\[ \Delta x = 10^{-10} \text{ m} \]

but

\[ \Delta \text{PE} = \frac{1}{2} k \Delta x^2 \]

or

\[ 1.6 \times 10^{-19} \text{ J} = \frac{1}{2} k (10^{-10} \text{ m})^2 \]

\[ k \approx \frac{3.2 \times 10^{-19} \text{ J}}{10^{-20} \text{ m}^2} \]

\[ k = 32 \frac{\text{N}}{\text{m}} \]
Problem 4. The harmonic oscillator (continued)

b) You are given a Schrödinger Equation with the following potential:

\[ V(x) = \frac{1}{2} ma^2 x^2 - bx \]

You could go to work trying to solve the Schrödinger Equation for this new potential, or, you could save yourself lots of time and effort by noticing something useful. You can rewrite this potential in the following form:

\[ V(x) = \frac{1}{2} ma^2 (x - x_0)^2 \]

To do so, you must be willing to shift your energy scale and you must also relate the offset position \( x_0 \) to the original parameter, \( b \). Show how the required energy shift and offset position are related to \( b \). (10 points)

\[
V(x) = \frac{1}{2} m\omega_0^2 \left[ x^2 - 2x x_0 + x_0^2 \right] \\
= \frac{1}{2} m\omega_0^2 x^2 - m\omega_0^2 x x_0 + \frac{1}{2} m\omega_0^2 x_0^2
\]

By this type of rewrite, you see that any additional linear term, like \(-bx\), is just satisfying that you have a harmonic one with a shifted equilibrium position.

\[
\text{required energy shift} = \frac{1}{2} b^2 / m
\]

c) For the traditional harmonic oscillator with potential energy of \( \frac{1}{2} ma^2 x^2 \) we expect a ground state wave function of the form, \( \psi_0(x) = Ae^{-\frac{m\omega_0 x^2}{2\hbar}} \). Write down the ground state wavefunction you expect for the potential in part (b). Explain your reasoning. (10 pts)

All we have is harmonic one around \( x = x_0 \), instead of \( x = 0 \). So define a new \( x' = x - x_0 \) and you expect

\[
\psi(x') = Ae^{-\frac{m\omega_0 (x-x_0)^2}{2\hbar}} \\
\]

or \( Ae^{-\frac{m\omega_0 x^2}{2\hbar}} \)