3. Review: Complex Numbers

(based on notes by M. Dubson and S. Pollack, modified by A. Becker)

Wave functions are in general complex functions. Here is a short review of complex numbers:

(i) \( i = \sqrt{-1} \Rightarrow i \cdot i = -1 \Rightarrow i = -1/i \Rightarrow 1/i = -i \)

(ii) Any complex number \( z \) can be written as

\[ z = x + iy \quad \text{(Cartesian form)} \]

or \( z = A e^{i\theta} \quad \text{(Polar form)} \)

You can visualize \( z \) by thinking of it as a point in the complex plane. The above formula yield to

\[ \text{Re}[z] = x = A \cos \theta \quad , \quad \text{Im}[z] = y = A \sin \theta \]

using Euler's relation \( e^{i\theta} = \cos \theta + i \sin \theta \)

(the latter can be proven with a Taylor series expansion)

(iii) The complex conjugate of \( z \) is \( \bar{z} = x - iy = A e^{-i\theta} \)

Any complex number can be turned into its complex conjugate by replacing every \( i \) with \(-i\).

(iv) Note that

\[ |z|^2 = z \cdot \bar{z} = (x + iy)(x - iy) = x^2 + iy x - i y x + y^2 = x^2 + y^2 \]

\[ = z \cdot \bar{z} = A e^{i\theta} A e^{-i\theta} = A^2 e^{i\theta-i\theta} = A^2 = x^2 + y^2 \]

is a purely real number. We call \( |z| = \sqrt{x^2 + y^2} = A \) the modulus of \( z \) or amplitude of \( z \), while
\( \theta \) is called the phase of \( z \).

Important: \( z^2 \neq z \cdot z \neq |z|^2 = z \cdot z^* \)

\[ \text{in general} \]

Squaring complex numbers does not always yield a real result.

(v) A useful fact, used above already:

\[ e^{i \theta_1 + i \theta_2} = e^{i \theta_1} \cdot e^{i \theta_2} \quad \text{or} \quad e^{i(\theta_1 + \theta_2)} = e^{i \theta_1} e^{i \theta_2} \]