Physics 3210 Spring 2008

Problem Set 11

NOTE: PROBLEMS 2–4 SHOULD BE WORKED IN ORDER. ALSO, THERE ARE NO LONG CALCULATIONS IN ANY OF THESE PROBLEMS.

1. Here is a simple problem that shows explicitly that $L$ doesn’t lie along $\omega$ in general. Note that the results are valid at a particular instant in time, but that the time average of $L$ will in fact be in the direction of $\omega$.

(a) Consider a point mass $m$ traveling in a circle of radius $a$ (in the $xy$-plane) with angular velocity $\omega = (0, 0, \omega_3)$. Find the angular momentum vector $L = r \times p$ with respect to the origin.

(b) Repeat part (a) but with the circle parallel to the $xy$-plane at a height $z = z_0$.

(c) Repeat part (b) but with two equal but diametrically opposite masses.

(d) Repeat parts (a)–(c) but using $L = I \cdot \omega$.

2. We have seen that the rate of change of an arbitrary vector $A$ as seen in the lab (or inertial) frame is related to its change as seen in a rotating (or body) frame by the equation

$$
\left( \frac{dA}{dt} \right)_{\text{lab}} = \left( \frac{dA}{dt} \right)_{\text{rot}} + \omega \times A \tag{1}
$$

where $\omega$ is the instantaneous angular velocity vector of the rotating frame as seen in the lab frame.

In this problem, you will work out some simple practical consequences of this result. In all of the sections below, be sure to show and explain every step in your derivations. Most of them are very short.

(a) Let $r$ be the coordinate vector of a moving particle. Show

$$
\left( \frac{d^2 r}{dt^2} \right)_{\text{lab}} = \left( \frac{d^2 r}{dt^2} \right)_{\text{rot}} + 2\omega \times \left( \frac{dr}{dt} \right)_{\text{rot}} + \frac{d\omega}{dt} \times r + \omega \times (\omega \times r). \tag{2}
$$

(b) We now want to take into account the general case where the rotating body is also in translational motion with respect to the lab. Consider an inertial (lab) coordinate system with axes $(x_0, y_0, z_0)$ and a moving (and in general rotating) body coordinate system with axes $(x, y, z)$. Let the origin of the $(x, y, z)$ coordinates be located at the instantaneous point $R$ with respect to the $(x_0, y_0, z_0)$ axes. If a particle has the coordinates $r_0$ in the inertial frame and $r$ in the body-fixed frame, then these coordinates are related by (see the figure below)

$$
r_0 = R + r.
$$
Show that this leads to the generalized expression

$$\frac{d^2 r_0}{dt^2}_{\text{lab}} = \frac{d^2 R}{dt^2}_{\text{lab}} + \frac{d^2 r}{dt^2}_{\text{rot}} + 2 \omega \times \frac{d r}{dt}_{\text{rot}} + \frac{d \omega}{dt} \times r + \omega \times (\omega \times r).$$

(c) Newton’s second law for a single particle in the inertial frame gives

$$m \left(\frac{d^2 r_0}{dt^2}\right)_{\text{lab}} = F^{(\text{ext})}$$

where $F^{(\text{ext})}$ is the external force. Show that for an observer fixed in the frame that is translating and rotating we have

$$m \left(\frac{d^2 r}{dt^2}\right)_{\text{rot}} = F^{(\text{ext})} - m \left(\frac{d^2 R}{dt^2}\right)_{\text{lab}} - 2 m \omega \times \frac{d r}{dt}_{\text{rot}} - m \omega \times (\omega \times r) - m \frac{d \omega}{dt} \times r.$$ 

This shows that the rotation produces several extra terms that act like forces. (These are called fictitious forces.) The term $-m (d^2 R/dt^2)_{\text{lab}}$ arises from the acceleration of the body system as seen in the lab frame, and may be called a translation force. The term $-2 m \omega \times (dr/dt)_{\text{rot}}$ is called the Coriolis force, and it vanishes unless the particle moves in the rotating frame along a direction different from $\hat{\omega}$. On the other hand, the centrifugal force $-m \omega \times (\omega \times r)$ acts even on a stationary particle. Finally, the azimuthal force $-m (d\omega/dt) \times r$ occurs only for a coordinate system with angular acceleration.

(d) Consider a person standing motionless on a carousel that rotates in the $xy$-plane with angular velocity $\omega = \omega \hat{z}$. Explain the magnitude and direction of the centrifugal force.
(e) Consider a person standing motionless on the earth at a polar angle $\theta$ as shown below. This person will feel an effective gravitational force $mg_{\text{eff}}$ due to the actual gravitational force $mg$ plus the centrifugal force $F_{\text{cent}}$ as shown. Using $6.4 \times 10^6$ m for the earth’s radius, calculate the percent correction to $g$ at the equator due to the centrifugal force. What is the correction at the poles?

(f) For what value of $\theta$ will the angle $\phi$ between $g$ and $g_{\text{eff}}$ be a maximum?

(g) Consider again the carousel from part (d) rotating at a constant velocity $\omega = \omega \hat{z}$. Suppose that a person is walking radially inward with speed $v$ relative to the carousel and at a distance $r$ from the center. Show that in order for the person to walk a straight radial line, he must supply a tangential friction force equal in magnitude to the Coriolis force, and that this friction force supplies the torque necessary to account for the change in angular momentum of the person as he walks.

(h) Now consider someone walking tangentially on the carousel, in the direction of the carousel’s motion, with speed $v$ relative to the carousel and at a constant radius $r$. Let $V = \omega r$ be the speed of a point on the carousel at radius $r$ as observed in the lab frame. Then relative to the lab frame the person is walking in a circle of radius $r$ at speed $v + V$, and hence he has an acceleration $(v + V)^2/r$ relative to the lab. This acceleration is caused by an inward pointing friction force at the
person’s feet, so

$$F_{\text{friction}} = \frac{m(v + V)^2}{r} = \frac{mv^2}{r} + \frac{mV^2}{r} + \frac{2mvV}{r}.$$ 

Explain the meaning of the three terms in this expression.

3. A ball is dropped from a height $h$ above the earth at a polar angle $\theta$ (measured down from the north pole). How far to the east is the ball deflected by the time it hits the ground? Solve this to first order in the Coriolis force (in other words, neglect the effect of the Coriolis force on the vertical speed of the ball).

4. (Bonus) Consider a cone rolling on a table. Let the origin of the cone be its tip, and let this point be fixed in the lab frame. The instantaneous $\omega$ for the cone is its line of contact with the table, because these are the points that are instantaneously at rest. This line precesses around the origin with an angular velocity $\Omega$. Note that $\omega$ only changes its direction, not its magnitude. See the figure below.

Consider the special case of a point $P$ that lies a distance $r$ along the instantaneous $\omega$ and which is motionless with respect to the cone. Let $Q$ be the point on the axis of the cone that lies directly above $P$.

(a) Show that the only fictitious force acting on $P$ is the azimuthal force.

(b) Find the acceleration of $P$. [*Hint:* Note that $Q$ is a distance $y = r\tan\beta$ above $P$. Let $P'$ be the point directly below $Q$ an infinitesimal time $dt$ later. Show that to a very good approximation the point $P$ is now a distance $h = (1/2)(\omega^2 y)(dt)^2$ above the table.]

(c) Calculate the azimuthal force on a mass $m$ located at $P$ and show that it is consistent with part (b). [*Hint:* What is the magnitude and direction of $\Omega$? Then what is the magnitude and direction of $d\omega/dt$? Now what is the magnitude and direction of the azimuthal force $F_{az}$?]

5. (This problem is based on the free symmetric top treated on pages 45–52 of the notes. See also Taylor, Problems 10.43 and 10.46.)

Suppose I throw a uniform circular disc ($I_1 = I_2 = I$) into the air (but neglecting gravity) so that it spins with angular velocity $\omega$ about an axis that makes an angle $\alpha$ with respect to the axis ($\hat{x}_3$) of the disc.

(a) Show that the magnitude $\omega$ of the angular velocity $\omega$ is a constant.
(b) Show that to me (i.e., the lab frame) $\hat{x}_3$ precesses around $L$ with frequency

$$\tilde{\omega} = \frac{\omega}{7} \sqrt{(l^2 - I_3^2) \sin^2 \alpha + I_3^2} = \omega \sqrt{4 - 3 \sin^2 \alpha}.$$

6. (Taylor, Problem 10.47). Pretend that the earth is a smooth, uniform, spherical rock of mass $M$ and radius $R = 4000$ mi. And it’s spinning about its usual axis (through the North Pole) at its usual rate (i.e., angular velocity $\omega$). Now suppose that a mountain of mass $10^{-8}M$ is added at colatitude (i.e., polar angle) $60^\circ$. This causes the earth to begin free precession. How long will it take for the North Pole to move 100 mi. from its current position? [Hint: Because of the mountain, the earth has only one axis of symmetry $\hat{x}_3$ and it passes through the mountain. The only somewhat tricky part of this problem is to evaluate the principal moments. Use some physical intuition and note that the size of the mountain is small compared to the size of the earth.]

7. Let $T \in L(V)$ and let $V_\lambda = \{v \in V : Tv = \lambda v\}$.

(a) Show $V_\lambda$ is a $T$-invariant subspace of $V$. (This means $v \in V_\lambda$ implies $Tv \in V_\lambda$. See Problem Set 4, problem 10.)

(b) Show $V_\lambda = \text{Ker}(T - \lambda I)$. [Hint: If $A, B$ are sets, then to show $A \subset B$ and $B \subset A$ it suffices to show that $A \cap B$ and $B \cap A$.]

8. An operator $T \in L(V)$ with the property that $T^n = 0$ for some $n \in \mathbb{Z}^+$ is said to be nilpotent. Show that the only eigenvalue of a nilpotent operator is 0.

9. If $S, T \in L(V)$, show (carefully, clearly and completely) that $ST$ and $TS$ have the same eigenvalues. [Hint: First use the corollary on page 132 of the LA Part 1 notes and the theorem on page 1 of the LA Part 2 notes to show that 0 is an eigenvalue of $ST$ if and only if 0 is an eigenvalue of $TS$. Now assume $\lambda \neq 0$, and let $ST(v) = \lambda v$. Show that $Tv$ is an eigenvector of $TS$.]

10. (a) Consider the rotation operator $R(\alpha) \in L(\mathbb{R}^2)$ defined by

$$R(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}.$$ 

Does $R(\alpha)$ have any eigenvectors? Explain.

(b) Repeat part (a) but now consider rotations in $\mathbb{R}^3$.

11. Consider the following matrices:

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -1 \\ 13 & -3 \end{bmatrix}$$

(a) For each matrix, find all eigenvalues and linearly independent eigenvectors over $\mathbb{R}$. 

5
(b) For each matrix, find all eigenvalues and linearly independent eigenvectors over $\mathbb{C}$.

(c) If either $A$ and/or $B$ is diagonalizable, find the transition matrix and put $A$ and/or $B$ into diagonal form.