Name (Please print legibly)  

1 Instructions

Your exam will look similar to this study guide, though it won’t be as long. The exam is closed book and closed notes, but the last page of the exam will contain relevant equations. Tear off the page if you desire.

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2 Preliminary Questions

1. Consider a particle of mass $m$ moving in a conservative potential $U(\vec{r})$, where the particle is located in space at the point specified by $\vec{r}$

(a) (4 points) Write the Lagrangian for the system in terms of $\vec{r}$, $\dot{\vec{r}}$, and $U$.

**Solution:**
\[
\mathcal{L} = \frac{1}{2}m(\dot{\vec{r}})^2 - U(\vec{r}) \quad \text{I’m testing if you know the general form for the kinetic energy of a particle.}
\]

(b) (4 points) Write this Lagrangian using spherical polar coordinates as generalized coordinates.

**Solution:**
\[
\mathcal{L} = \frac{1}{2}m \left( \dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2 \right) - U(r, \theta, \phi) \quad \text{I’m testing if you can convert from a general expression for the kinetic energy into a particular coordinate system.}
\]

(c) (8 points) Find the Lagrange equations of motion

**Solution:**
\[
\begin{align*}
r: \quad & m \left( r\ddot{\theta}^2 + r^2\sin^2(\theta)\ddot{\phi}^2 - \dot{r} \right) - \frac{\partial U}{\partial r} = 0 \\
\theta: \quad & m \left( r^2\sin(\theta)\cos(\theta)\ddot{\phi}^2 - 2\dot{r}\dot{\theta} - r^2\ddot{\theta} \right) - \frac{\partial U}{\partial \theta} = 0 \\
\phi: \quad & m \left( 2\dot{r}\sin^2(\theta)\dot{\phi} + 2r^2\sin(\theta)\cos(\theta)\dot{\theta}\dot{\phi} + r^2\sin^2(\theta)\ddot{\phi} \right) + \frac{\partial U}{\partial \phi} = 0
\end{align*}
\]

Here, I’m interested if you can correctly evaluate the $(d/dt)(\partial L/\partial \dot{q}_i)$ term. That evaluation requires the correct application of chain rule and product rule. It also requires that you understand the distinction between partial and total differentiation.
Consider a cylinder of radius $R$, mass $m$ and moment of inertia $I$, rolling without slipping on an inclined plane with angle $\alpha$ relative to the horizontal ($x$-axis). The plane sits in a gravitational field with acceleration $-g\hat{y}$. Neglect motion in the $z$-dimension.

(a) (4 points) Choose (a) generalized coordinate/coordinates and write the Lagrangian using your generalized coordinates. State clearly what coordinates you are choosing and what name you’ve given them.

**Solution:** The motion is 1-D, but there are several possible choices for the generalized coordinate. I’ll pick distance along the inclined plane and call that coordinate $w$

$U = mgy = mgw\sin(\alpha)$

$T = \frac{1}{2} \left[ mx^2 + y^2 \right] + I\dot{\phi}^2 = \frac{1}{2} \left[ mw^2 + I\dot{w}^2/R^2 \right]$  

$L = T - U$

I’m interested if you recognize that the problem only needs one generalized coordinate, and if you can make a sensible choice for that coordinate. The angle of rotation of the cylinder $\phi$ is also a perfectly good choice.

(b) (4 points) Find the Lagrange equation(s) of motion for your coordinate(s).

**Solution:**

$-mg\sin(\alpha) - \ddot{w}(m + I/R^2) = 0$

Here, I expect you to be able to correctly evaluate $T$ and $U$ in your choice of generalized coordinates.

(c) (4 points) Solve the equation(s) of motion.

**Solution:**

$w(t) = -\frac{g}{2} \frac{\sin(\alpha)}{1 + I/mR^2} t^2 + Ct + D,$

where $C$ and $D$ are constants of integration

I’m interested to see if you know that an equation of motion can be solved for a dynamical variable as a function of time. I chose a simpler example where you can solve the 2nd order differential equation just by integrating twice.
3. Consider a cylinder of radius $R$, mass $m$ and moment of inertia $I$, rolling without slipping on an inclined plane with angle $\alpha$ relative to the horizontal ($x$-axis). The plane sits in a gravitational field with acceleration $-g\hat{y}$. Neglect motion in the $z$-dimension. Choose redundant generalized coordinates $w$ and $\phi$.

(a) (4 points) $w$ and $\phi$ are not independent write the equation of constraint.

Solution:

\[ R\dot{\phi} - w = 0 \]

Do you know what an equation of constraint is?

(b) (2 points) Find the Lagrangian in terms of $w$ and $\phi$.

Solution:

\[ \frac{1}{2} \left[ mw^2 + I\dot{\phi}^2 \right] - mgw \sin(\alpha) \]

Can you evaluate $T$ and $U$ in gen. coords?

(c) (6 points) Find the modified Lagrange equations of motion.

Solution:

\[ -mg \sin(\alpha) - mw = \lambda \]
\[ I\ddot{\phi} = \lambda R \]

Do you understand how to work with the formula I provide to find modified Lagrange equations of motion?

(d) (10 points) Solve the equation of motion for $\phi$ and $w$, and determine $\lambda$.

Solution:

\[ w(t) = -\frac{g}{2} \left( \frac{\sin(\alpha)}{1 + I/mR^2} \right) t^2 + Ct + D \]
\[ \phi(t) = -\frac{g}{2R} \left( \frac{\sin(\alpha)}{1 + I/mR^2} \right) t^2 + Ct/R + D/R \]
\[ \lambda = -gm \sin(\alpha) \frac{I}{mR^2 + I} \]

I’m interested to see if you know that a system of equations of motion can be solved for a several dynamical variables as a function of time. Again the math is relatively simple but the idea is important.
4. A particle of mass $m$ is constrained to move in a plane. It is connected to a point in the plane by a spring with spring constant $k$.

(a) (4 points) Write the Lagrangian choosing coordinates that will permit you to exploit any symmetry in the problem.

**Solution:** Clearly problem is symmetric in the angle of rotation about a line that passes through the spring attachment and is perpendicular to the plane of motion. Let’s use plane polar coordinates $(r, \phi)$ centered on this point.

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + (r\dot{\phi})^2) - \frac{1}{2}kr^2$$

Can you choose generalized coordinates to take advantage of symmetry? If I’d given this on a real exam I would have told you the $U = kr^2/2$. I just forgot in this study guide.

(b) (8 points) Find the Lagrange equations of motion in your choice of variables. If possible, use any ignorable coordinate to find first integrals of motion.

**Solution:**

$$r : -m\ddot{r} + mr\ddot{\phi}^2 - kr = 0$$

$$\phi : mr^2\ddot{\phi} = l,$$

where $l$ is a constant.

Do you know what an ignorable coordinate and first integral of motion are?

(c) (4 points) Rewrite the equations of motion using the first integrals of motion to eliminate the generalized velocities of any ignorable coordinates.

**Solution:** For the $r$-equation, we can eliminate $\dot{\phi}$ in preference of $l$.

$$r : -m\ddot{r} + l^2/mr^3 - kr = 0$$

Do you know how to use an ignorable coordinate to eliminate the associated generalized velocity?

(d) (4 points) Use the equation of motion in the previous part to find the static solution.

**Solution:**

$$r_0 = \left( \frac{l^2}{mK} \right)^{\frac{1}{4}}$$

Can you find static solutions?
(e) (8 points) Determine if the static solution is stable or unstable. If the static solution is stable, find the frequency of small oscillations about the static solution.

**Solution:** The solution is stable.

\[ \omega = 2 \sqrt{\frac{k}{m}} \]

Can you solve linearized equations of motion?

(f) (4 points) Find the energy function \( h \).

**Solution:**

\[ h = \frac{1}{2} m (\dot{r}^2 + (r \dot{\phi})^2) + \frac{1}{2} kr^2 \]

Do you recognize that \( h = E \) because the transformation from cartesian to generalized coordinates didn’t involve time? If you do you can save a lot of time, because you don’t have to evaluate the formula for \( h \).

(g) (4 points) Is \( h \) conserved? Is \( h = E \)?

**Solution:**

\[ \frac{\partial L}{\partial t} = 0 \Rightarrow h = C, \]

with \( C \) a constant. In addition, \( h = E \) because the equations of coordinate transformation \( x = r \cos \phi \) and \( y = r \sin \phi \) don’t depend explicitly on time.

Do you know the conditions when \( h \) is conserved and when \( h = E \)? Do you recognize \( T + U = E \)? Again, you could solve this problem just by evaluating given formulas but it’s much faster if know the conditions for \( h \) to be conserved and for \( h = E \).
3  Equation table

You may find the following equations helpful.

3.1  Equations pertaining to the Lagrangian formulation

\[ \mathcal{L} = T - U, \]
where \( T \) is the kinetic energy, and \( U \) is the potential energy.

\[ \frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = 0, \]
for a minimum set of generalized coordinates \( q_i \).

\[ \frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = -\sum_{j=1}^{P} \lambda_j \frac{\partial G_j}{\partial q_i}, \]
where \( q_i \) are a redundant set of \( N \) generalized coordinates, related by \( P \) equations of constraint of the form \( G_j(q_i) = 0 \). Here, \( i \) indexes the coordinates and \( j \) the equations of constraint.

The energy function \( h \) has the following properties.

\[ h = \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \]
\[ \frac{dh}{dt} = -\frac{\partial \mathcal{L}}{\partial t} \]

3.2  Equations pertaining to coordinate transformations.

Plane polar coordinates.

\[ x = r \cos(\phi) \]
\[ y = r \sin(\phi) \]

Spherical polar coordinates, with \( \phi \) the azimuthal angle and \( \theta \) the polar angle.

\[ x = r \sin(\theta) \cos(\phi) \]
\[ y = r \sin(\theta) \sin(\phi) \]
\[ z = r \cos(\theta) \]

Cylindrical coordinates, with \( \rho \) the radial coordinate.

\[ x = \rho \cos(\phi) \]
\[ y = \rho \sin(\phi) \]
\[ z = z \]
3.3 General mechanics

The kinetic energy of an object rotating with angle $\phi$ about an axis is

$$T = \frac{1}{2}I\dot{\phi}^2,$$

where $I$ is the momentum of inertia about the axis of rotation.

3.4 Other formulae

Taylor expansion of a function $f(x)$ about $x_0$.

$$f(x) \approx f(x_0) + \frac{df(x_0)}{dx}(x - x_0) + \frac{1}{2}\frac{d^2f(x_0)}{dx^2}(x - x_0)^2 + \ldots$$